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JANUARY 1948

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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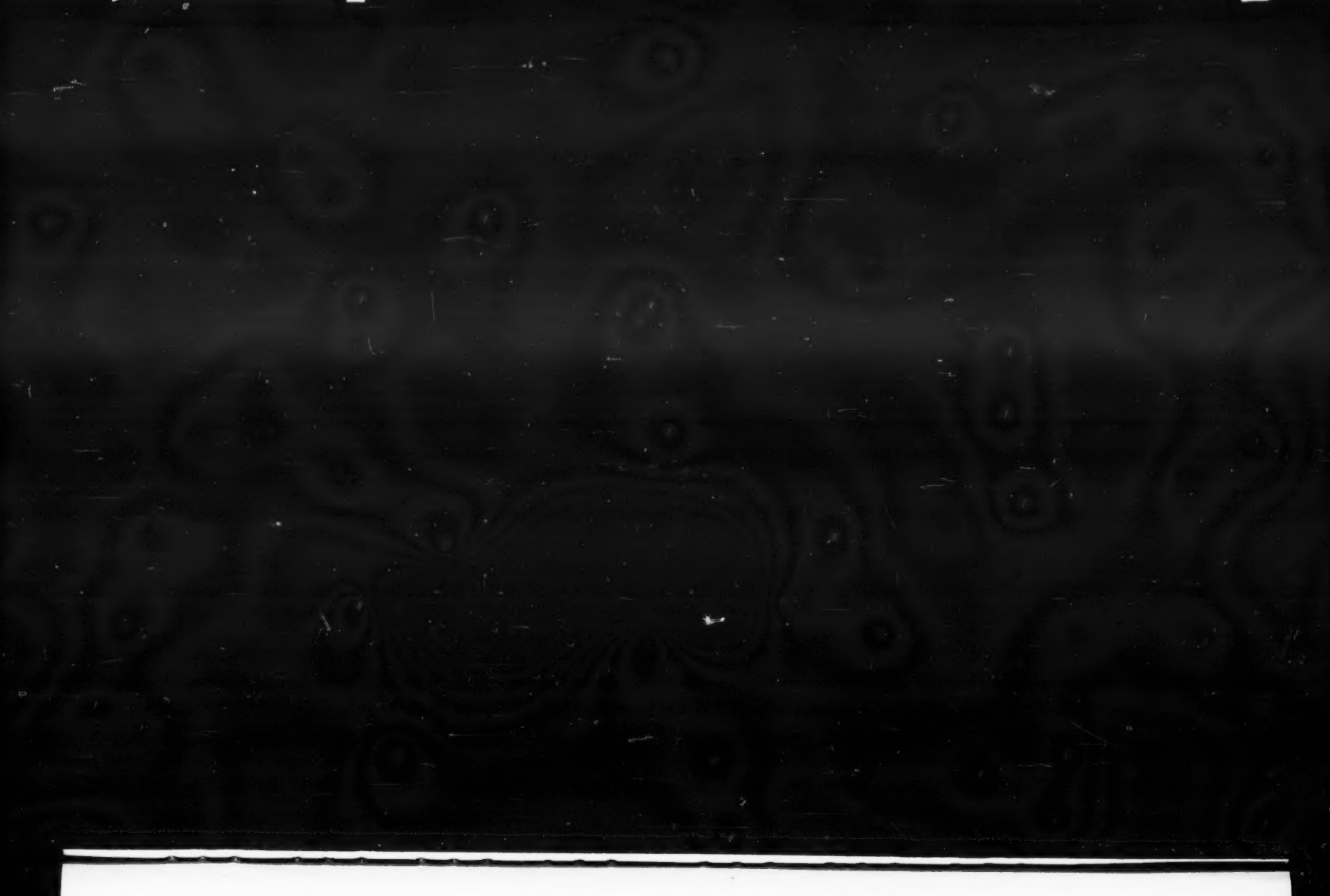

THE MATHEMATICS PROGRAM IN THE COLLEGE OF THE UNIVERSITY OF CHICAGO*

E. P. NORTHROP, College of the University of Chicago

1. **Introduction.** Some three years ago I was given the opportunity to address members of the Association on the subject of the mathematics program in the College of the University of Chicago.† That report was made when the staff concerned with the program was small, when the program itself was hardly more than a year old, and when the aims of the program were only dimly seen. The staff has since grown in number from three to twelve or fourteen, the program is about to enter its fifth year, and the aims have become clearer—although they still constitute the chief battleground over which the staff struggles, week by week, in staff meetings. It is hoped that the present report on work which has been accomplished in the intervening years and on plans being made for the future, with some indication of areas of success and failure, and some information about what has been done to meet the failures, will be of interest to teachers of mathematics and perhaps even of use to those who are planning or are already engaged in similar enterprises elsewhere.

2. **The Chicago program of liberal education.** The aims of the College mathematics program at Chicago are intimately bound with those of the College program as a whole. Stated briefly, the College is engaged in presenting a four-year program of liberal education. On the assumption that the faculty is a better judge than the student of what constitutes a liberal education, the program is a required one, common for all students. And on the assumption that the liberally educated person should have learned something about each of the great areas of human thought, the student is required to show competence in the knowledge, methods, and skills relevant to the humanities (including English and a foreign language), the social sciences, the biological and physical sciences, and mathematics.

The student is admitted to the College on the basis of psychological tests and tests in reading and writing skills, to which it is hoped that there will soon be added a test in quantitative skills. He may apply for admission after two, three, or four years of high school, or after a year of college elsewhere. Once admitted, he is faced with the problem of passing fourteen six-hour comprehensive examinations covering the general areas mentioned above. From some of them he may be excused on the basis of placement tests which he takes directly following his admission; upon completing the others he is awarded the Bachelor of Arts degree. He normally prepares for four of the examinations each year, although he may take any of them at any time he feels he is adequately prepared for them.

In such a program there is no problem of designing and presenting several

* Presented at the summer meeting of the Mathematical Association of America, New Haven, Connecticut, September 2, 1947.

† Mathematics in a Liberal Education, this MONTHLY, vol. 52 (1945), pp. 132–137.

courses in mathematics at the first-year college level: one—presumably terminal—for students of the humanities, another for physical science majors, a third for engineering students, and so on. This fact appears to me to constitute a distinct advantage. For I am convinced of the desirability of a single course at the level concerned, and deplore current attempts to construct different courses for different purposes. A liberal education is presumably the kind of education everyone ought to have, regardless of the career he expects to enter. If this be true, then any mathematics course designed as an integral part of such an education should be one that every student ought to take, whether or not he plans to take further courses in mathematics. This is one of the assumptions on which the Chicago program is based, and the greater part of the remainder of this paper will consist of a description of the kind of course in mathematics which has emerged from this assumption and concomitant assumptions.

Actually, two one-year courses will be discussed. The first, Mathematics 1, is a required part of the College program. It is ordinarily taken in the first year of the College, and serves as a springboard for a three-year sequence in the natural sciences. The second, Mathematics 2, is designed for those students who wish to pursue the subject further. It is not an elective in the sense that it may be used to satisfy the requirements for the bachelor's degree: these requirements may be met only by passing the fourteen comprehensive examinations mentioned above, one of which is Mathematics 1. Students who wish to make plans for specialized work in one area or another, however, are permitted to take advanced courses for which they are qualified concurrently with the specified courses in the College program.

3. **Mathematics 1, the required general course.** What ought to go into a one-year course in mathematics to be taken by *all* students—students who, it may be assumed, have learned something of the ABC's of the subject in their two or more years of high school algebra and geometry? This question can be answered only after a prior question has been discussed: What, if anything, in mathematics is appropriate to a liberal education? I raised and attempted to answer this question in my earlier report. Rather than to repeat myself here, I should like to quote from W. L. Schaaf, who came close to my convictions in the matter ten years ago when he wrote, in the pages of this MONTHLY:*

> The distinctive feature of mathematics, one that sets it apart from all other domains of human achievement, is that it exemplifies, or, more strictly, it is, a ***unique*** style of thinking. . . . The uniqueness of mathematics as a mode of thought results from the following features: (1) the formulation of generalizations, (2) the method of postulational thinking, and (3) the ceaseless quest for greater rigor.

From this view of the place of mathematics in liberal education one may conclude that the student should be taught to think deductively, to know what a deductive system is, to understand the relation between an abstract deductive system and its models, or concrete interpretations, and to have some apprecia-

* Vol. 44 (1937), p. 447.

tion of what rigor is and how it may be achieved. Add to this the fact that he should learn, both for the skills themselves and for the part they will play in his later courses in science, how to understand and to deal with problems of quantity and space, and the content of a year course in mathematics appropriate to a program of liberal education becomes fairly clear: it should include at least the study of logic, algebra, and geometry. As the staff is now teaching Mathematics 1, about one fifth of the course is devoted to logic, two fifths to algebra, and two fifths to analytic geometry and trigonometry.

In the earlier versions of our course the approach to logic was Aristotelian, but we soon found—and it is not surprising that we did so—that this approach is inadequate even for the elementary mathematics we attempt to develop. We defended it at the time on the grounds that many of our students were deathly afraid of symbolism in any form and needed the reassurances to be gained by emphasizing such everyday phenomena as verbal propositions and the syllogism. Upon cautiously experimenting with fundamental ideas from modern symbolic logic, however, we discovered that, properly introduced to them, the student can take in his stride variables and constants, propositions and relations between them, propositional functions and quantifiers, sets and relations between them, undefined terms and rules of definition, and unproved propositions and rules of proof. This revelation has prompted us to push even further in a comprehensive revision of our text which is now being prepared by the staff for the academic year 1948–49.*

This excursion into the fundamental concepts of modern symbolic logic culminates in a careful study of the commutative group as an example of a deductive system. The group was chosen in preference to other possible examples, such as the theory of relations, or the propositional calculus, or certain simple geometric systems, for a number of reasons. The group is adequate in that it exemplifies almost everything which has gone before: sets, operations, relations, quantifiers, undefined terms, definitions, postulates, theorems, proofs, and models. It is simple in that the undefined terms and postulates are few in number, and the proofs easy to follow. It is meaningful in that the postulates and theorems find immediate application in the work of the course which follows. Finally its models are both numerous and significant, and offer excellent opportunities to study the relation between an abstract system and its concrete interpretations.

* In outline form, this revision will include the following topics: 1. Theory of Classes (general concept of class or set, universe, null class, subsets, sets of ordered pairs, complement, union, intersection, relations between classes, variables); 2. Relation and Function (relation as a set of ordered pairs, as a correspondence, as a table, function, unary and binary operations); 3. Propositions (general concept, negation, conjunction, disjunction, implication, equivalence); 4. Propositional Functions and Quantifiers (propositional functions of one variable, sentential conjunctions of propositional functions, universal and existential quantifiers, negation, sets and propositional functions, propositional functions of two variables, mixed quantifiers); 5. Propositional Calculus (symbolism, truth tables and universal propositions, special universal propositions); 6. Deductive systems (undefined terms, definitions, postulates and theorems, simple example of a deductive system, model of a deductive system, rules of proof, applications to simple proofs, indirect proof.)

It may appear that I have dwelt overlong on the material which occupies only the first fifth of the year. But it is primarily this material, together with its later applications, which makes Mathematics 1 unique among first-year college courses, and which the staff has found most difficult to organize and present. There is no question in our minds of its importance as groundwork for the rest of the course. For it is here that the student first learns the rudiments of scientific discourse, and without clear ideas about deductive systems he will gain little from the discussions of real numbers and analytic geometry, which are later developed as such systems.

The second part of the course has already been referred to as "algebra," although it is more algebra in the elementary textbook sense than in the strict sense, being a combination of the study of real numbers and real-valued functions. The development of real numbers, which is postulational in nature, may be described briefly as follows: the postulates for addition and multiplication, for subtraction, for division, the order postulates, and the continuity postulate are introduced successively, with pauses for assimilation of the postulates and their resulting theorems between them. It is assumed—and properly, we have found—that the student is sufficiently aware of the common properties of natural numbers, integers, and rational and real numbers to make them available for purposes of motivation and exemplification, although other models of the system at each stage of its development are also introduced and discussed. It should be apparent that this approach to real numbers offers ample opportunity to spend what time may be necessary in drill work on the elementary operations. The complex numbers are constructed from the reals (this development gives the student a view of the alternative, constructive approach to number systems) and their properties discussed briefly.

In that part of the "algebra" which is strictly analytic, the notions of real-valued variable and function are easily grasped by the student after the introduction he has had to the general concepts of variable and function in the first part of the course. Here the work continues with the construction of functional relations from given data, the use of functional notation, a variety of classifications of functions, and the solutions of equations as zeros of functions.

In the light of its experience the staff proposes to institute a number of changes in the approach to both real numbers and functions. In the first place, it seems desirable to have the student spend more time on the study of groups, and to exploit his increased power in that area by working into the real numbers through, successively, the additive group, the multiplicative group, and the field. Again, once the field has been achieved, it appears well to introduce the notion of function—integral and rational—and to say something here about the solution of inequalities, which are studied in connection with the order postulates that follow. The rest of the analytic material of the so-called "algebra" section could then be properly postponed until the study of analytic geometry.

Originally, the course next devoted a few weeks to the study of plane geometry. Analytic geometry was then developed in essentially the historical manner

as a combination of algebra and geometry. Having brought the work in logic and algebra to a satisfactorily respectable degree of rigor, however, the staff found, somewhat to its dismay, that it could not achieve that same degree of rigor in synthetic plane geometry without getting into involved and time-consuming considerations which promised to be irksome to both student and instructor. After a vain search for a solution to this problem, it was tentatively decided that the student's previous training in geometry could be counted upon for intuitive interpretations of geometric concepts and that geometry might best be constructed from scratch in the modern analytic fashion—by defining point as an ordered pair of real numbers, line as a set of points satisfying a linear equation, and so on. It is intended to explore this approach further in the proposed revision, although some members of the staff are conducting what may amount to a filibuster to convince the rest of their colleagues of the desirability of an approach through vectors. In any event, it seems unlikely that the study of analytic geometry will be carried much beyond a comprehensive treatment of the straight line and circle. In the future, as now, the remaining few weeks of the course will probably be devoted to the introduction and study of the trigonometric functions and their properties. It should be emphasized, however, that the treatment of trigonometry is one of which the analyst, and not the surveyor, would approve.

The majority of the students in the College take no mathematics beyond the course which has just been described. For such students it may be thought that the course is inadequate in its failure to include a number of topics for which various authors have argued with varying degrees of fervor. Although I cannot agree with such authors as the one who would include number theory and permutations so that the student will be able to face later in life the problems he will find, respectively, in current popular magazines and in trying to arrange his dinner guests, I am quite ready to admit the desirability of giving the terminal student some idea of such topics as the calculus and statistical analysis. The difficulty, of course, is that of finding time for acceptable developments of these topics in a year course which meets four hours per week. The tendency at Chicago is to sacrifice breadth of coverage of many topics for depth of understanding of a few fundamental ones. The choice of those which constitute Mathematics 1 is defended on the grounds that any student who has successfully fulfilled the course requirements will be able to pursue, independently and with understanding, any further topics in mathematics in which he may have a genuine but nonprofessional interest.

4. **Mathematics 2, the optional continuing course.** It is apparent that Chicago students who desire to continue their formal work in mathematics need some instruction designed to bridge the gap between the somewhat unconventional work of Mathematics 1 and the more advanced courses in calculus and post-calculus subjects offered by the Department of Mathematics in the (graduate) Division of the Physical Sciences. The College has accepted the responsibility of providing such continuing instruction in Mathematics 2. In terms of

both time and number of students, the staff has had less experience with this course than with Mathematics 1,* but its experience is sufficient for the formulation of some tentative conclusions about what can and cannot be done with students who have met the established prerequisite for entrance into the course, namely, a grade of middle C or higher in Mathematics 1.

Briefly described, the first third of Mathematics 2 is devoted to an extension of some of the material, initiated in Mathematics 1, of college algebra, trigonometry, and analytic geometry; and the last two thirds to the calculus and to further work in analytic geometry. This description is disarming in the sense that the topics considered are more sophisticated, and the approach more rigorous, than is ordinarily the case in courses in the subjects named. Thus, for example, not only are the postulates which characterize the real numbers reviewed, with careful considerations of such new concepts as absolute value and the related inequalities, but the integers as well are characterized, with careful considerations of alternative formulations of the principle of finite induction and a wealth of applications of the same. Again, abstract vector spaces are introduced and studied in connection with polar coordinates and the polar representation of complex numbers. The study of the calculus is conducted on a similarly mature level. Here the emphasis is on a clear understanding of and ability to work with the concepts and properties of function, limit, continuity, derivative, and definite integral. It is no small satisfaction to the staff, for example, to find that students of the age of high school seniors can be taught to understand clearly the relations between existence of a limit, continuity, and differentiability.

Although the staff hopes ultimately to have as complete a text for Mathematics 2 as it has for Mathematics 1,† it has at present only a set of notes, amounting to an extended outline, for the work of the first part of the course. Notes of this kind lend themselves easily to yearly revision, and can be expanded to textbook form after ideas concerning the most effective material to include in the course have crystallized. The remainder of the course at present consists of adaptations of material from Randolph and Kac's excellent text, *Analytic Geometry and Calculus*, the only serious drawback of which, for our purposes, is the fact that it was not written as an actual sequel to the material which precedes it in our two-year program.

5. **Conclusion.** Here, then, for whatever information, encouragement, and use it may be to teachers of mathematics, is a brief description of the present state of the mathematics program at the College of the University of Chicago. After four years, difficulties and shortcomings still plague us, and will probably continue to do so for years to come. But I am convinced that our enthusiastic staff, having already demonstrated how much more can be done with the ele-

* Mathematics 1 is now (1947–48) in its fifth year and is being taken by some 450 students; Mathematics 2 is in its second year, and is being taken by some 75 students.

† The text for Mathematics 1, prepared by the author of this paper with the assistance of other members of the College mathematics staff, is Fundamental Mathematics, Chicago, University of Chicago Bookstore, 1946 (2nd ed.)

mentary student than is ordinarily expected of him, will ultimately achieve a program which will be meaningful both in the sense of liberal education and in the sense of mathematical education. Indeed, in my first report I argued boldly for the meaningfulness of our program within the framework of a liberal education, and somewhat more timidly for its meaningfulness for the continuing student. I have since heard from a surprising number of mathematicians who feel that the program is a fine one for the continuing student, but much too difficult for the purposes of liberal education. This reaction I should like to regard both as an indication of vigor in the health of our program and as a confirmation of the thesis that one course for all students, not different courses for different groups of students, is appropriate to the level of instruction concerned.

AN ANALLAGMATIC CUBIC

C. E. NOBLE, Neenah, Wisconsin

1. Introduction. This paper is devoted to the study of the cubic

$$LX(Y^2+Z^2)+MY(Z^2+X^2)+NZ(X^2+Y^2)+KXYZ=0, \quad \text{or} \tag{1}$$
$$\sum LX(Y^2+Z^2)+KXYZ=0.$$

The cubic (1) is invariant under the isogonal transformation,

$$x':y':z'=\frac{1}{x}:\frac{1}{y}:\frac{1}{z},$$

a quadratic transformation of the plane, which sends the point P having trilinear coördinates x, y, z into the point P' having trilinear coördinates $1/x$, $1/y$, $1/z$ [1]. Cubics invariant under the isogonal transformation are often called anallagmatic cubics [2, 5]. Neuberg [5] stated in 1923 that he was not aware of any geometric interpretation of the cubic (1). In 1930 Goormaghtigh [3] presented a geometric interpretation of a special case of cubic (1). To the writer's knowledge, no additional information concerning this cubic has appeared in the literature. In this discussion a number of geometric interpretations of cubic (1) are presented together with many of its properties.

2. Geometric interpretations. We shall find it convenient to refer to the cubic (1) as the *plus-sign* cubic to distinguish it from the *minus-sign* cubic

$$\sum LX(Y^2-Z^2)=0, \tag{2}$$

the other anallagmatic cubic in the triangle. The cubic (2) has been widely discussed in the literature [2, 5].

Throughout this paper, P and P' represent two isogonally conjugate points having trilinear coördinates x, y, z and $1/x$, $1/y$, $1/z$, respectively. The angles,

the lengths of the sides, and the area of the reference triangle are denoted by A, B, C; a, b, c; and S, respectively. The points of intersection of the lines AP, BP, CP and AP', BP', CP' with the opposite sides of the reference triangle are designated by the letters A_p, B_p, C_p and A_p', B_p', C_p', respectively. Generally, the coördinates x_1, y_1, z_1 of a point Q are considered to be proportional to the distances of Q from the sides of the reference triangle. If we insist that x_1, y_1, z_1 represent the actual distances of Q from the sides of the reference triangle, then x_1, y_1, z_1 satisfy the relation $2S=ax_1+by_1+cz_1$ and are called *actual coördinates*.

Several geometric interpretations of the plus-sign cubic will now be given as theorems, beginning with the special case derived by Goormaghtigh [3].

THEOREM 1. *If the areas of triangles $A_pB_pC_p$ and $A_p'B_p'C_p'$ are equal, then the locus of P and P' is a plus-sign or minus-sign cubic.*

If x, y, z are the actual coördinates of P, then the actual coördinates of A_p' A_p', and so on, are respectively $[0, 2Sy/(by+cz), 2Sz/(by+cz)]$, $[0, 2Sz/(bz+cy), 2Sy/(bz+cy)]$, and so on. The conditions that the areas of triangle $A_pB_pC_p$ and $A_p'B_p'C_p'$ be equal is that

$$(3) \qquad (by + cz)(cz + ax)(ax + by) = \pm (bz + cy)(cx + az)(ay + bx).$$

If the plus sign is taken on the right, equation (3) becomes the equation of the minus-sign cubic

$$(4) \qquad \sum a(b^2 - c^2)x(y^2 - z^2) = 0.$$

If the minus sign on the right is taken, the equation (3) becomes

$$(5) \qquad \sum a(b^2 + c^2)x(y^2 + z^2) + 4abcxyz = 0,$$

which is the equation of the special plus-sign cubic obtained by Goormaghtigh. This special case of the plus-sign cubic appears to be the only one in the literature with the exception of the degenerate locus made up of a line and a conic which are isogonally conjugate, namely

$$(lx + my + nz)(lyz + mzx + nxy) = 0, \quad \text{or}$$
$$\sum mnx(y^2 + z^2) + (l^2 + m^2 + n^2)xyz = 0.$$

In a similar manner it may be shown that if the areas of triangles $A_pB_p'C_p'$ and $A_p'B_pC_p$, $A_p'B_pC_p'$ and $A_pB_p'C_p$, or $A_p'B_p'C_p$ and $A_pB_pC_p'$ are required to be equal, then the locus of P and P' is a plus-sign or minus-sign cubic. The locus of P and P' is also a plus-sign or minus-sign cubic if we equate the areas of triangles $\overline{A}_p\overline{B}_p'\overline{C}_p'$ and $\overline{A}_p'\overline{B}_p\overline{C}_p$, $\overline{A}_p'\overline{B}_p\overline{C}_p'$ and $\overline{A}_p\overline{B}_p'\overline{C}_p$, or $\overline{A}_p'\overline{B}_p'\overline{C}_p$ and $\overline{A}_p\overline{B}_p\overline{C}_p'$, where $\overline{A}_p$ and $\overline{A}_p'$ are the harmonic conjugates of A_p and A_p' with respect to B and C, and $\overline{B}_p$, $\overline{B}_p'$, $\overline{C}_p$ and $\overline{C}_p'$ are analogous points.

THEOREM 2. *If the six point conic $A_pB_pC_pA_p'B_p'C_p'$ passes through a seventh point, then the locus of P and P' is a plus-sign cubic.*

Through the six points A_p, B_p, C_p, A_p', B_p', C_p' passes a conic whose equation is

$$\sum x(y^2+z^2)YZ - xyz(X^2+Y^2+Z^2) = 0, \tag{6}$$

where X, Y, Z are the running coördinates. If this system of conics is required to pass through a fixed point, say $Q(X_1, Y_1, Z_1)$, then the locus of the points P and P' is the plus-sign cubic

$$\sum Y_1Z_1x(y^2+z^2) - (X_1^2+Y_1^2+Z_1^2)xyz = 0. \tag{7}$$

It is interesting to observe that if the cubic (7) is forced to pass through the incenter (1, 1, 1) of triangle ABC, then the fixed point Q must lie on the inscribed conic

$$X^2+Y^2+Z^2-2YZ-2ZX-2XY = 0.$$

Similarly, if the cubic (7) passes through an excenter, say (1, 1, −1), then the point Q must lie on the inscribed conic

$$X^2+Y^2+Z^2+2YZ+2ZX-2XY = 0.$$

THEOREM 3. *If the six point conic $A_pB_pC_pA_p'B_p'C_p'$ be a rectangular hyperbola, then the locus of P and P' is a plus-sign cubic.*

The condition on the coefficients of the conic (6) in order that it be a rectangular hyperbola is

$$\sum \cos A\, x(y^2+z^2) + 3xyz = 0, \tag{8}$$

which shows that the locus of P and P' is then a plus-sign cubic.

Similarly, a plus-sign cubic is obtained as the locus of P and P' if the six point conic $\overline{A}_p\overline{A}_p'B_pB_p'C_pC_p'$ (or $A_p\overline{A}_p'\overline{B}_pB_p'C_pC_p'$, or $A_pA_p'B_pB_p'\overline{C}_p\overline{C}_p'$) is required either to pass through a seventh point or to be a rectangular hyperbola.

The conic passing through $\overline{A}_p$, $\overline{A}_p'$, $\overline{B}_p$, $\overline{B}_p'$, $\overline{C}_p$, $\overline{C}_p'$ degenerates into the trilinear polars of P and P' with respect to the triangle ABC. These trilinear polars are, respectively,

$$yzX + zxY + xyZ = 0 \quad \text{and} \quad xX + yY + zZ = 0.$$

If these two lines are perpendicular, then the locus of the points P and P' is the plus-sign cubic

$$\sum \cos A\, x(y^2+z^2) - 3xyz = 0. \tag{9}$$

If these two lines meet in a point on the line

$$LX + MY + NZ = 0. \tag{10}$$

the locus of P and P' is the minus-sign cubic

$$\sum Lx(y^2-z^2) = 0. \tag{2}$$

If the line (10) is the infinite line, we have the condition that the trilinear polars of P and P' be parallel, in which case the locus of P and P' is

$$\sum \sin A x(y^2 - z^2) = 0. \tag{11}$$

THEOREM 4. *The locus of the foci of inscribed conics tangent to a given line is a plus-sign cubic.*

Casey [1] shows that the equation of a conic inscribed in triangle ABC with the point $F(x, y, z)$ as focus may be expressed in the form

$$\sin A_1(X/x)^{1/2} + \sin B_1(Y/y)^{1/2} + \sin C_1(Z/z)^{1/2} = 0, \tag{12}$$

where A_1, B_1, C_1 are respectively the angles subtended from the point F by the sides BC, CA, AB of triangle ABC.

Since $\sin A_1 = x2S/[(x^2+z^2+2xz \cos B)(x^2+y^2+2xy \cos C)]^{1/2}2R$, where R is the circumradius of ABC, and $\sin B_1$ and $\sin C_1$ are equal to similar quantities, equation (12) may be written as

$$\sum [x(y^2 + z^2 + 2yz \cos A)X]^{1/2} = 0. \tag{13}$$

The condition that the inscribed conic (13) be tangent to the line

$$\frac{X}{L} + \frac{Y}{M} + \frac{Z}{N} = 0 \tag{14}$$

is that

$$\sum Lx(y^2 + z^2) + 2(L \cos A + M \cos B + N \cos C)xyz = 0. \tag{15}$$

Thus, the locus of the focus F is a plus-sign cubic. Since the foci of inscribed conics are isogonally conjugate points, the other focus $F'(1/x, 1/y, 1/z)$ of the conic (13) also lies on the cubic (15).

THEOREM 5. *The locus of the foci of inscribed conics whose conjugate axes are of constant length is a plus-sign cubic.*

If the actual coördinates of the focus F of a conic inscribed in triangle ABC are x, y, z, then the actual coördinates of the other focus F' are $(2Syz/U, 2Szx/U, 2Sxy/U)$, where $U = ayz+bzx+cxy$. It is generally known that the square of the semi-conjugate axis of a conic is equal to the product of the distances of the foci of the conic to any line tangent to the conic [6]. Taking one of the sides of triangle ABC as the tangent to the inscribed conic and $1/k$ as the square of the semi-conjugate axis, we obtain the relation

$$xyz2S/(ayz + bzx + cxy) = 1/k. \tag{16}$$

Introducing $(ax+by+cz)/2S$ into equation (16) in order to make it homogeneous, we get the equation of the locus of the foci of a family of inscribed conics whose conjugate axes are constant to be

$$\sum bcx(y^2 + z^2) + (a^2 + b^2 + c^2 - 4S^2k)xyz = 0. \tag{17}$$

THEOREM 6. *The locus of the foci of inscribed conics whose centers lie on a given line is a plus-sign cubic.*

If the coördinates of the foci F and F' of an inscribed conic are the same as given above, then the coördinates of the midpoint F'' of the line segment FF' are x_1, y_1, z_1, where

$$x_1 = \frac{2axyz + x^2(bz + cy) + yz(by + cz)}{2U}.$$

If we require F'', which is the center of the inscribed conic having the points F and F' as foci, to lie on the line

$$LX + MY + NZ = 0,$$

then the foci F and F' travel over the locus

$$\sum (cM + bN)x(y^2 + z^2) + 2(aL + bM + cN)xyz = 0. \tag{18}$$

If the center F'' is required to lie on the line

$$\begin{aligned} Xa(-aL + bM + cN) + Yb(aL - bM + cN) \\ + Zc(aL + bM - cN) = 0, \end{aligned} \tag{19}$$

then the locus of the foci F and F' is the cubic (15). Thus, when an inscribed conic remains tangent to the line (14), the locus of its foci is the cubic (15), and the locus of its center is the line (19).

THEOREM 7. *If the locus of the point of intersection of the tangents at P and P' to the circumconic passing through P and P' be a straight line, then the locus of P and P' is a plus-sign cubic.*

The equation of the circumconic passing through the points $P(x, y, z)$ and $P'(1/x, 1/y, 1/z)$ is

$$\sum x(y^2 - z^2)YZ = 0.$$

The tangents to this conic at the points P and P' meet in the point T whose coördinates are $x(y^2+z^2)$, $y(z^2+x^2)$, $z(x^2+y^2)$. If T lies on the line

$$LX + MY + NZ = 0,$$

then P and P' lie on a plus-sign cubic whose equation is

$$\sum Lx(y^2 + z^2) = 0. \tag{20}$$

THEOREM 8. *If P and P' are conjugate points with respect to a given conic, the locus of P and P' is a plus-sign cubic.*

The condition that the point $P'(1/x, 1/y, 1/z)$ lie on the first polar of the point $P(x, y, z)$ with respect to the general conic

$$DX^2 + EY^2 + FZ^2 + 2LYZ + 2MZX + 2NXY = 0$$

is the plus-sign cubic

$$\sum Lx(y^2+z^2)+Kxyz=0, \tag{1}$$

where $K=D+E+F$.

3. Other properties. It is readily seen that the equation (1) is invariant under the isogonal transformation. The cubic curve represented by equation (1) passes through the vertices of the reference triangle ABC and meets the sides of triangle ABC in the points in which the line

$$\frac{X}{L}+\frac{Y}{M}+\frac{Z}{N}=0 \tag{21}$$

cuts these sides. Since this cubic is invariant under the isogonal transformation, it follows that it is tangent to the isogonal conjugate of line (21) at the vertices; that is, it is tangent at the vertices to the circumconic

$$\frac{YZ}{L}+\frac{ZX}{M}+\frac{XY}{N}=0.$$

In fact, the cubic is tangent at the vertices to any curve whose isogonal conjugate passes through the points in which the line (21) cuts the sides of triangle ABC.

The polars of $P(x, y, z)$ with respect to the conics

$$\begin{gathered} KX^2+2LYZ+2MZX+2NXY=0, \\ X^2-Y^2=0, \qquad Y^2-Z^2=0 \end{gathered} \tag{22}$$

are, respectively,

$$\begin{gathered} X(Kx+Mz+Ny)+Y(Lz+Nx)+Z(Ly+Mx)=0, \\ Xx-Yy=0, \qquad Yy-Zz=0. \end{gathered} \tag{23}$$

These three polars meet in a point, say $\overline{P}$, if

$$\sum Lx(y^2+z^2)+Kxyz=0. \tag{1}$$

The cubic (1) is called the Jacobian of the three conics (22) [4]. It may readily be shown that the cubic (1) is the Jacobian of any three conics, not passing through the same four points, of the family

$$KX^2+2LYZ+2MZX+2NXY+\mu(X^2-Y^2)+\nu(Y^2-Z^2)=0. \tag{24}$$

The coördinates of $\overline{P}$, the point of intersection of the polars (23), are $1/x$, $1/y$, $1/z$. Hence, $\overline{P}$ lies on the Jacobian (1) and is identical with P'. The points P and $\overline{P}$ are called conjugate points on the Jacobian. Thus, the isogonally conjugate points P and P' on the cubic (1) are conjugate points on this cubic considered as the Jacobian of three conics of the family (24). This property of the cubic leads to the following theorem.

THEOREM 9. *If the points P and P', Q and Q' are pairs of isogonally conjugate points on the cubic* (1), *then the points of intersection of the lines PQ and $P'Q'$, PQ' and $P'Q$ are isogonally conjugate points on this cubic.*

This theorem results from the fact that if two pairs of opposite vertices, P and P', Q and Q', of a quadrilateral are conjugate points with respect to a conic, then the third pair, R and R', are also conjugate points with respect to the conic [4]. Hence, if we are given three pairs of isogonally conjugate points on the cubic (1), we may, in general, construct an infinite number of isogonally conjugate points on the cubic by use of the ruler only.

If we allow the point Q to approach the point P along the cubic, then the point Q' approaches the point P' along the cubic. In the limit the following result is obtained.

THEOREM 10. *The tangents to the cubic* (1) *at the two isogonally conjugate points P and P' on the cubic meet in a point R which lies on the cubic, and the line PP' meets the cubic in the point R', the isogonal conjugate of R.*

We may easily show that the circumconic which passes through any two isogonally conjugate points P and P' on the cubic (1) meets the cubic a sixth time in the point R which is the point of intersection of the tangents to the cubic at the points P and P'.

In general, the cubic (1) is non-singular. However, it has a double point at the incenter of triangle ABC when it passes through this point. For, if the cubic (1) passes through the incenter, it is necessary that $K+2L+2M+2N=0$. This is the condition that the cubic have a double point at the incenter. Similarly, if cubic (1) passes through any one of the excenters of triangle ABC, it possesses a double point at that particular excenter.

For the special case in which $|L| = |M| = |N|$ in equation (1), we see that $(0, M, -N)$, $(L, 0, -N)$, $(L, -M, 0)$ are inflection points of (1). No generality is lost by taking $L=M=N=1$. For this case, the polar conic of the inflection point $(1, -1, 0)$ is

$$X^2 - Y^2 + Z(X - Y)(K - 2) = 0.$$

This is a degenerate conic made up of the tangent to the cubic at the inflection point and the harmonic polar of the inflection point with respect to the cubic. Their equations are, respectively,

$$X + Y + Z(K - 2) = 0 \quad \text{and} \quad X - Y = 0.$$

Analogous results may be produced for the inflection points $(0, 1, -1)$ and $(1, 0, -1)$.

If the entire configuration is subjected to the isogonal transformation, the cubic and the harmonic polars remain invariant. However, the tangent lines transform into circumscribed conics each of which meets the cubic four times at a vertex. This follows from the fact that the tangents to the cubic at the inflection points each meet the cubic three times at an inflection point which

lies on the sides of triangle ABC. If $K=0$ in (1), then the centers of the degenerate conics mentioned above are located at the incenter of triangle ABC, and each of the circumscribed conics which meet (1) four times at a vertex passes through the incenter.

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WHAT ARE SET FUNCTIONS?

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1. **Point functions and set functions.** In speaking of a function, one usually thinks of a *point function* $f(x)$: A rule is given by which, in a certain domain, to each number x (or, more generally, to each point x of a certain space) there correspond numbers $f(x)$. If in this statement we replace x by a *set* X of numbers or points x, we come immediately to the more general notion of a *set function* $f(X)$; that is: A rule is given by which to each set X, belonging to a certain family of sets of a space, there correspond numbers $f(X)$. In the most important case of a single-valued set function, to each set X only one number $f(X)$ is attached.

Particular instances of set functions have been well known to mathematicians for a long time, even since antiquity. We have only to remember the length of an interval or of an arc, the area of a plane region or of a surface, the volume of a solid, the content or the measure of a set. Of course, the definite integral is also a very important example of a set function. In fact, the integral $\int_a^b f(x)\,dx$ is an interval function; to be specific, it is a function of the interval $[a, b]$. In the case of multiple integrals one integrates over regions. The idea of more general integration over (measurable) sets is due to H. Lebesgue [1]. In fact, it is Lebesgue [2] who may be considered as the founder of the theory of set functions; since his original studies the theory has been further developed by many other mathematicians. We now know that such a general theory is interesting and useful, furnishing a solid, unifying basis for more special theories, and is a valuable source for many applications. A systematic presentation and

discussion of set functions will be found in the forthcoming book of H. Hahn and A. Rosenthal [3].

Further generalizations of basic notions are possible, as for the range of values of set functions. Instead of numbers one can take points of a space (for instance, of Banach spaces) or, even more generally, sets as values of the function. In the case of this last generalization, where sets correspond to sets, one arrives at the notion of mapping one space into another space. We shall not go so far here, but shall consider only single-valued set functions whose range of values consists of real numbers (including $+\infty$ and $-\infty$).

The notion of set functions becomes fruitful by the introduction of some specific properties, such as additivity.

2. **Additive set functions.** Let $\mathfrak{M}$ be a system of sets in a quite general space, and let the set function $\phi(X)$ be defined for each set X that is a member of $\mathfrak{M}$, denoted symbolically by $X\epsilon\mathfrak{M}$. From the first we shall exclude the case that $\phi(X)=+\infty$ for *all* $X\epsilon\mathfrak{M}$ or that $\phi(X)=-\infty$ for *all* $X\epsilon\mathfrak{M}$. Then ϕ is called *additive* in $\mathfrak{M}$ if for any two *disjoint* sets $A\epsilon\mathfrak{M}$ and $B\epsilon\mathfrak{M}$, for which also their sum $(A+B)\epsilon\mathfrak{M}$, we have

$$\phi(A+B)=\phi(A)+\phi(B). \tag{1}$$

Here $\phi(A)$ and $\phi(B)$ cannot have infinite values of different signs, since otherwise the right side of (1) would be meaningless.

EXAMPLES: Let $\mathfrak{M}$ be the system of all subsets of a given infinite set E.

(*a*) For every $A\epsilon\mathfrak{M}$ let $\phi(A)$ be the number of elements of A. (In particular, let $\phi(A)=+\infty$ if A is an infinite set.)

(*b*) For every $A\epsilon\mathfrak{M}$ let $\phi(A)$ be one plus the number of elements of A. (In particular, let $\phi(A)=+\infty$ again if A is an infinite set.)

(*c*) Let $\phi(A)=0$ if A is finite, and $\phi(A)=+\infty$ if A is infinite.

Then ϕ is additive in the Examples (*a*) and (*c*), while ϕ is non-additive in the Example (*b*).

Let ϕ be additive in $\mathfrak{M}$, and let all sets discussed belong to $\mathfrak{M}$. In studying ϕ it will be suitable to assume that $\mathfrak{M}$ is a *field;* that is, with any two sets $A\epsilon\mathfrak{M}$ and $B\epsilon\mathfrak{M}$, their sum $A+B$ and their difference $A-B$ also belong to $\mathfrak{M}$. It can easily be seen also that their intersection $A\cdot B$ belongs to $\mathfrak{M}$.

Of course, by induction, we obtain immediately from (1):

$$\phi(A_1+A_2+\cdots+A_n)=\phi(A_1)+\phi(A_2)+\cdots+\phi(A_n),$$

provided that the sets A_i, $(i=1, 2, \cdots, n)$, are disjoint.

For two sets which are not necessarily disjoint one obtains:

$$\phi(A)+\phi(B)=\phi(A+B)+\phi(A\cdot B). \tag{2}$$

The demonstration of this follows from the fact that $A=A\cdot B+(A-B)$ and $B=A\cdot B+(B-A)$; thus $\phi(A)=\phi(A\cdot B)+\phi(A-B)$ and $\phi(B)=\phi(A\cdot B)+\phi(B-A)$. Hence, by addition, $\phi(A)+\phi(B)=\phi(A\cdot B)+[\phi(A\cdot B)+\phi(A-B)+\phi(B-A)]=\phi(A\cdot B)+\phi(A+B)$.

3. **Totally additive set functions.** The set function ϕ, additive in $\mathfrak{M}$, is called *totally additive* (or *completely additive*) in $\mathfrak{M}$ if for every sequence of *disjoint* sets $A_\nu \in \mathfrak{M}(\nu = 1, 2, \cdots)$, for which also their sum $\underset{\nu}{S} A_\nu \in \mathfrak{M}$, we have

$$\phi(\underset{\nu}{S} A_\nu) = \sum_\nu \phi(A_\nu). \tag{3}$$

In Example (a) of §2, ϕ is a totally additive set function. But, in Example (c), ϕ is an additive set function which is *not* totally additive; for if $B = \underset{\nu}{S} b_\nu$ is any denumerable subset of E, we have $\phi(b_\nu) = 0$, but $\phi(B) = +\infty$.

More significant examples are the Lebesgue measure, which is totally additive in the system of measurable sets, and the Jordan content which is additive, but not totally additive, in the system of the sets measurable in the sense of Jordan.

For the study of totally additive set functions, it is natural to assume $\mathfrak{M}$ to be a *σ-field*, that is, a field which along with each sequence of sets $\{A_\nu\}$ also contains their sum $\underset{\nu}{S} A_\nu$. Now let ϕ be totally additive in the σ-field $\mathfrak{M}$, and let all sets discussed belong to $\mathfrak{M}$. Then the following theorems can easily be proved:

THEOREM (a). Let $\{A_\nu\}$ be a monotone increasing sequence of sets (that is $A_{\nu_1} \subseteq A_{\nu_2}$ for $\nu_1 \leqq \nu_2$), then $\phi(\underset{\nu}{S} A_\nu) = \lim_\nu \phi(A_\nu)$.

THEOREM (b). Let $\{A_\nu\}$ be a monotone decreasing sequence of sets (that is, $A_{\nu_1} \supseteq A_{\nu_2}$ for $\nu_1 \leqq \nu_2$). If not all $\phi(A_\nu)$ are infinite, then $\phi(\underset{\nu}{D} A_\nu) = \lim_\nu \phi(A_\nu)$, where $\underset{\nu}{D} A_\nu$ designates the intersection of the sequence $\{A_\nu\}$.

THEOREM (c). If ϕ is *monotone increasing* (that is, $A \subseteq B$ implies $\phi(A) \leqq \phi(B)$), then we have for every sequence $\{A_\nu\}$ of sets

$$\phi(\underset{\nu}{S} A_\nu) \leqq \sum_\nu \phi(A_\nu). \tag{4}$$

The following important result is due to H. Hahn [4]:

THEOREM (d). The set function ϕ attains the absolute maximum and the absolute minimum of its values.

From (d) it follows immediately that ϕ is bounded either above or below (since ϕ cannot attain both the values $+\infty$ and $-\infty$); and, moreover, if ϕ is finite, then ϕ is also bounded.

We can reduce ϕ to monotone increasing functions by defining the *positive-function* ϕ^+ and the *negative-function* ϕ^- of ϕ in the following way.

For $A \in \mathfrak{M}$, we set

$$\phi^+(A) = \sup \phi(X) \quad \text{and} \quad \phi^-(A) = -\inf \phi(X), \tag{5}$$

where X designates those subsets of A which belong to $\mathfrak{M}$. Then we have

$$\phi = \phi^+ - \phi^-. \tag{6}$$

Besides, the *absolute-function* $\bar{\phi}$ is defined by

$$\bar{\phi} = \phi^+ + \phi^-. \tag{7}$$

The functions ϕ^+, ϕ^-, $\bar{\phi}$ are monotone increasing and, with ϕ, they are also totally additive. The reduction to monotone increasing functions, given by (6), is very useful for the proofs of many theorems.

4. Zero-sets for ϕ. A set $A \in \mathfrak{M}$ is called a *zero-set* for ϕ if $\phi(X) = 0$ for all subsets X of A which belong to $\mathfrak{M}$. If ϕ is totally additive in the σ-field $\mathfrak{M}$, then the system of all zero-sets for ϕ is also a σ-field.

$\mathfrak{M}$ is called *complete* for ϕ if every subset of each zero-set for ϕ is contained in $\mathfrak{M}$.

A given field (or σ-field) can be extended into a complete field (or σ-field). More specifically, if ϕ is an additive set function in the field $\mathfrak{M}$, then there is a field $\mathfrak{M}^0 \supseteq \mathfrak{M}$ and an extension of ϕ into a set function ϕ^0 which is additive in $\mathfrak{M}^0$ such that $\mathfrak{M}^0$ is complete for ϕ^0. In this statement the words "field" and "additive" can be replaced by the words "σ-field" and "totally additive."

5. ψ-continuous set functions. Let ϕ and ψ be two totally additive set functions in the σ-field $\mathfrak{M}$. Then ϕ is called *ψ-continuous* (or *totally continuous with respect to ψ*) if for every zero-set X for ψ we have also $\phi(X) = 0$; or, what amounts to the same thing, if every zero-set X for ψ is also a zero-set for ϕ. On the other hand, ϕ is called *purely ψ-discontinuous* if every set $M \in \mathfrak{M}$, with $\phi(M) \neq 0$, contains a zero-set X for ψ with $\phi(X) \neq 0$.

Example of a ψ-continuous function: Let μ be the one-dimensional Lebesgue measure. Then the Lebesgue integral $(M)\int f(x)dx$ (defined for the one-dimensional measurable sets M) is μ-continuous.

Example of a purely ψ-discontinuous function: The function ϕ defined in Example (a) of §2 is purely μ-discontinuous, if in the example E designates a linear infinite set and μ is again the one-dimensional Lebesgue measure.

It has been proved that a totally additive set function ϕ can be represented as the sum of a ψ-continuous and a purely ψ-discontinuous set function (both of which are also totally additive in $\mathfrak{M}$). If ϕ is finite, then there is only one such representation.

For the notion of the ψ-continuous set functions, the zero-sets for ψ play a distinguished role. A more general theory, also containing other interesting particular cases, can be developed if the sets S of any σ-field $\mathfrak{S} \subseteq \mathfrak{M}$ are distinguished as "singular sets," provided that every $X \in \mathfrak{M}$ which is a subset of S belongs also to $\mathfrak{S}$.

6. Measure. The general theory of measure, which contains the Lebesgue measure as a particular case, was founded by C. Carathéodory [5]. Let E be a given set, and consider the system $\mathfrak{E}$ of all subsets A of E. A set function $\phi(A)$, defined in $\mathfrak{E}$, is called a *measure function* if the following conditions are satisfied:

(a) For the empty set Λ we have: $\phi(\Lambda) = 0$.

(b) $\phi(A)$ is monotone increasing (*cf.* §3, (c)).

(c) $\phi(\sum_\nu A_\nu) \leqq \sum_\nu \phi(A_\nu)$.

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Because of (*a*) and (*b*), we always have: $\phi(A) \geqq 0$. The essential inequality (*c*) is the same as used above in (4).

A set $M \epsilon \mathfrak{E}$ is called *ϕ-measurable* if for M, together with *every* set $A \epsilon \mathfrak{E}$, we have

$$\phi(A) = \phi(A \cdot M) + \phi(A - M). \tag{8}$$

The ϕ-measurable sets M form a σ-field $\mathfrak{M}$, and ϕ is totally additive in $\mathfrak{M}$. This gives the relation between the measure functions (which, in general, are not additive) and the totally additive set functions.

7. **Integration.** The integral of a point function $f(x)$ can be defined with respect to any totally additive set function ϕ. Of course, the model for a general theory of integration is the Lebesgue integral. H. Lebesgue [1] performed the integration with respect to the Lebesgue measure in an n-dimensional Euclidean space R_n. Generalizing the Lebesgue integral and, simultaneously, also the Stieltjes integral, J. Radon [6] replaced the Lebesgue measure by an arbitrary totally additive set function. He still took the space R_n as the basis; but then M. Fréchet [7], referring to J. Radon, easily generalized his definition for abstract spaces.

Subsequently, such definitions and theories of integration have been developed in many ways by quite a few mathematicians. Perhaps the most simple and most fascinating definition of general integration is due to H. Hahn [8], and it has been extensively discussed in the above-mentioned book [3].

Let $\mathfrak{M}$ be a σ-field and let ϕ be a totally additive set function in $\mathfrak{M}$. According to §4 it is no restriction to assume $\mathfrak{M}$ to be complete for ϕ, since $\mathfrak{M}$ may be replaced by the extended σ-field $\mathfrak{M}^0$, if it is necessary to do so. Let $A \epsilon \mathfrak{M}$; then the subsets $M \epsilon \mathfrak{M}$ of A form also a σ-field, say $\mathfrak{A}$. Moreover, let $f(x)$ be a point function defined on A.

The measurability of $f(x)$ on A can be defined in the same way as in Lebesgue's theory; that is, for every number y, the set of those points $x \epsilon A$ at which $f(x) > y$ has to belong to $\mathfrak{M}$.

Now let $f(x)$ be measurable on A and let $\phi(A)$ be finite. The function f is called *ϕ-integrable* on A if there is a set function $\lambda(M)$, which afterwards is said to be the *ϕ-integral* of f, defined in $\mathfrak{A}$ and satisfying the following two conditions:

(1) $\lambda(M)$ is totally additive in $\mathfrak{A}$.

(2) $\lambda(M)$ has the following mean-value property:

Let $M \epsilon \mathfrak{A}$ and $c' \leqq f(x) \leqq c''$ for all $x \epsilon M$; then $c'\phi(M) \leqq \lambda(M) \leqq c''\phi(M)$ if $\phi^-(M) = 0$, (that is, if ϕ is monotone increasing on M). $c''\phi(M) \leqq \lambda(M) \leqq c'\phi(M)$ if $\phi^+(M) = 0$, (that is, if ϕ is monotone decreasing on M). [If herein $c = \pm \infty$ and $\phi(M) = 0$, one has to set $c\phi(M) = 0$.]

If f is ϕ-integrable on A, then it can be proved that the value $\lambda(A)$ is uniquely determined by (1) and (2), and hence it can be designated by

$$\lambda(A) = (A) \int f d\phi. \tag{9}$$

On the basis of this definition the whole theory of integration can be developed, including the representation of the integral by means of Lebesgue sums (which H. Lebesgue [1] had used as the definition of his integral in the particular case that ϕ designates the Lebesgue measure in R_n). Moreover, the Radon-Nikodym theorem [6, 9] can be obtained, which gives the following characterization of the set functions that are ϕ-integrals: In order that a set function $\lambda(M)$ be a ϕ-integral, it is necessary and sufficient that $\lambda(M)$ be totally additive and ϕ-continuous.

8. **Differentiation.** The theory of differentiation of set functions, founded also by H. Lebesgue [2], encounters more difficulties.

Let E be a metric space, and let a be any point of E; we designate by $S_{a\rho}$ the sphere with center a and radius ρ. Let ϕ and ψ be totally additive set functions in a σ-field $\mathfrak{M}$ of subsets of E. Moreover, let $\mathfrak{M}$ satisfy the following condition: to every $\rho>0$ there is a (non-empty) set of $\mathfrak{M}$, contained in the sphere $S_{a\rho}$. We say the sequence $\{M_\nu\}$ of sets of $\mathfrak{M}$ converges to a if there is a sequence of positive numbers $\{\rho_\nu\}$ with $\rho_\nu \to 0$, such that $M_\nu \subseteq S_{a\rho_\nu}$.

If there is a sequence $\{M_\nu\}$ converging to a, such that the sequence

$$\left\{\frac{\phi(M_\nu)}{\psi(M_\nu)}\right\}$$

converges to the value d, then d is called a *derivate* of ϕ with respect to ψ at the point a on $\mathfrak{M}$. The greatest and smallest of such values give the *upper* and *lower derivate*, respectively, and, if both coincide, we have the *derivative* of ϕ with respect to ψ at the point a on $\mathfrak{M}$.

But these definitions are still too general and would not furnish a satisfactory theory of differentiation without further restrictions on the systems of sets applied. This was first observed by H. Lebesgue [2] who, therefore, operating in the Euclidean space R_n and using the n-dimensional Lebesgue measure μ for ψ, introduced the notion of *regularity* of sets. A family $\mathfrak{M}$ of measurable sets M was called *regular* by him if for every point a there exists a number $\zeta(a)>0$ with the following property: Let $M_\nu \in \mathfrak{M}$, $(\nu=1, 2, \cdots)$, be any sequence of sets converging to a and let S_ν be the smallest n-dimensional sphere containing M_ν. Then the following condition must be satisfied:

$$\frac{\mu(M_\nu)}{\mu(S_\nu)} \geqq \zeta(a).$$

In the theory of differentiation, developed by H. Lebesgue with the aid of the notion of regularity, the Vitali covering theorem played an essential role. Now in different ways, conditions formed after the pattern of the Vitali covering theorem can be stated, such that a general and satisfactory theory of differentiation of ϕ with respect to ψ is possible. It includes theorems of the following type. The derivative of ϕ with respect to ψ exists almost everywhere. The ψ-integral of this derivative, or of any derivate, furnishes the ψ-continuous part of

the primitive function ϕ. Since the theory of differentiation of set functions is rather complicated, we do not go into further details here. Those interested in this subject may be referred to the discussion in Chapter V of the above-mentioned forthcoming book [3].

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MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.

SOME INTERESTING SQUARES

P. A. PIZÁ, San Juan, Puerto Rico

I was about to send to the MONTHLY as a problem the following innocent looking proposal:

PROBLEM: Find squares of four or more digits (of which the last one is not zero), so that their last three digits, when reversed, form a cube.

The solution of this problem has led me to find a family of squares which have some remarkable properties that, in my opinion, deserve communication in this note.

Let $A^2 = \cdots abc$ be the required squares and let $cba = d^3$. As $c \neq 0$, d^3 can only take the five values

$$125, \quad 216, \quad 343, \quad 512, \quad 729.$$

Then A^2 is one of

$$\cdots 521, \cdots 612, \cdots 343, \cdots 215, \cdots 927,$$

where the dots stand for other digits.

As squares cannot end in the digits 2, 3 or 7, the second, third and fifth possibilities are immediately eliminated.

Every square in 5 is of the type

$$(10x + 5)^2 = 100x^2 + 100x + 25 = 100x(x + 1) + 25.$$

Consequently every square which ends in 5 must also end in 25, and this eliminates the fourth possibility.

Therefore the squares that we are seeking must be of the type

$$A^2 = 1000y + 521,$$

and these are the squares which have the interesting properties that motivate this note. They could be considered independently of the above problem, but the problem is inserted to show how I was led to treat them.

There is an infinite number of these squares, the first few being:

$$\begin{array}{lll} 39^2 = 1\ 521, & 461^2 = 212\ 521, & 789^2 = 622\ 521, \\ 211^2 = 44\ 521, & 539^2 = 290\ 521, & 961^2 = 923\ 521, \\ 289^2 = 83\ 521, & 711^2 = 505\ 521, & 1\ 039^2 = 1\ 079\ 521. \end{array}$$

Observe that A is the sum at any point of the series of summands

$$A = 39 + 172 + 78 + 172 + 78 + 172 + 78 + \cdots,$$

where, after 39, the numbers $172 = 4 \cdot 43$ and $78 = 2 \cdot 39$ alternate as summands. As a matter of fact this series may also be extended towards the left into negative numbers

$$A = 39 - 78 - 172 - 78 - 172 - \cdots,$$

whose squares are positive and all end in the prime number 521.

Note also that in $A^2 = 1000y + 521$, y is again the sum at any point of the series of summands

$$y = 1 + 43 + 39 + 3 \cdot 43 + 2 \cdot 39 + 5 \cdot 43 + 3 \cdot 39 + 7 \cdot 43 + 4 \cdot 39 + 9 \cdot 43 + 5 \cdot 39 + \cdots,$$

where each successive summand after 1 is alternately 43 multiplied by the consecutive odd numbers 1, 3, 5, $\cdots$, and 39 multiplied by the consecutive numbers 1, 2, 3, $\cdots$.

Hence as

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

and

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$$

are respectively the successive squares and triangular numbers, we can express

y in the form

$$y = 1 + 43n^2 + 39\,\frac{n(n \pm 1)}{2},$$

where n can be any non-negative integer.

Therefore A^2 is of the general type

$$\begin{aligned} A^2 &= 500(125n^2 \pm 39n + 2) + 521 \\ &= 62500n^2 \pm 19500n + 1521 \\ &= (250n \pm 39)^2. \end{aligned}$$

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

All material for this department should be sent directly to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania.

EDITORIAL NOTE: The following papers were among those presented before the twenty-ninth Summer Meeting of the Association, New Haven, Conn., September 1, 1947, under the general title: "How to Solve It," a Symposium on Mathematical Problems at the College Level.

PROBLEMS IN MECHANICS

J. L. SYNGE, Carnegie Institute of Technology

PROBLEM I: *Practical—Find a number by a standard procedure.* A pendulum consists of two equal uniform bars AB, BC smoothly jointed at B and suspended from A. The mass of each bar is m, and its length is $2a$. Find the normal periods for small oscillations in a vertical plane under gravity, in the form $2\pi k(a/g)^{1/2}$, where k is a numerical constant.

PROBLEM II: *Aesthetic—Existence or non-existence of something simple.* For a rigid body, in general motion, show that there is no point at rest. Show also that, in general, there is one point, and only one, with no acceleration.

PROBLEM III: *Puzzle—We won't let it get away unsolved.* For a system of particles, prove that the kinetic energy of motion relative to the mass center may be expressed in the form

$$T = \frac{1}{2m}\Sigma m_i m_j V_{ij}^2$$

where m is the total mass of the system, m_i the mass of a typical particle, V_{ij} the velocity of m_i relative to m_j, and the summation contains one term for each pair of particles.

In considering, attacking, and solving a problem, we might note the following things:

1. Motivation—why do it?
2. Classification—what is it?
3. Plan
4. Details
5. Mistakes

Let us get *classification* out of the way first, because it is the easiest to deal with. All our three problems belong to *mechanics*. The student will recognize at once that they all belong to *dynamics* and not to *statics*, but he should draw finer distinctions. He should recognize that the first problem belongs to dynamics proper (kinetics), or, more precisely, to the theory of vibrations. The other two belong to *kinematics*—the idea of force plays no part in them.

The question of motivation (to use the jargon of the psychologists) is very important. Why does anyone feel an inclination to solve problems such as these? (I leave out such sordid motives as solving problems for the sake of obtaining a good grade—we are thinking on a higher plane.) Some people are moved to solve problems in mechanics because of their practical implications, and some are moved by the pleasure of employing a technique over which they have mastery. But this is not the whole story, as I shall indicate later.

If I had to work up interest in the first problem, I would point out that this is a simple case of an eigen-value problem, and that eigen-value problems belong to an interesting and important domain of mathematics which goes over into quantum mechanics. Further, on the more practical side, the general question of natural frequencies is important to engineers, and, although the given problem is not perhaps of any immediate practical importance, it is typical of a class of problems which are.

However, I feel that if we go only so far we omit something very essential. Students of mathematics are human beings, and as such are susceptible to aesthetic feelings. If you know where to look for it, there is something aesthetic about this first problem. The double pendulum is not, in a sense, a complicated thing, but when you describe it and its motion in mathematical terms it does become rather complicated. Out of this complication, there emerges something very simple—a pair of numbers, representing the frequencies. Anyone will, I believe, experience some aesthetic emotion in obtaining something simple from something complicated. Without any claim to originality in this matter (although I do not recall previously seeing the statement so put), let me enunciate an "aesthetic principle" as follows:

$$\text{COMPLEXITY} \xrightarrow[\text{understanding}]{\text{Through intellectual}} \text{SIMPLICITY}$$

I believe that this aesthetic principle, properly interpreted, is valid not only for mathematics and science generally, but for the arts as well.

The aesthetic aspect is more pronounced in the second problem. Here we are indeed faced with something complicated and hard to visualize—the three-dimensional motions of a rigid body. But we are asked something simple. First,

is there any particle instantaneously at rest? Secondly, is there any particle which instantaneously has no acceleration? A further aesthetic charm of variety arises from the fact that we can answer the first question in a flash by considering the infinitesimal screw displacement of the body in infinitesimal time, but the second question forces us into analytical methods.

So much for motivation. As regards a plan of attack, it is easy enough for a mature and experienced person to lay one out. But it is a very hard thing for a student to do, and here he needs all encouragement. After he has solved a problem, he should discuss his solution (either with himself or someone else), and describe what steps he has carried out. Our first problem lends itself very well to planning, because it is of a standard type. Assuming that the student knows the Lagrangian method, the steps might be planned as follows:

1. Assign generalized coördinates.
2. Calculate the kinetic and potential energies approximately, each as a quadratic form.
3. Write down the equations of motion.
4. By a trigonometric or exponential substitution, obtain a determinantal equation for the frequencies.
5. Solve this equation. This will be easy, because the equation will be quadratic, there being two degrees of freedom.

The second problem hardly lends itself so well to detailed planning. In fact, overemphasis on planning would lead us at once into an analytic attack, and so hide from us the simple descriptive way of answering the first part of the problem. Thus planning is something that can be overdone—we must not become slaves to a routine, or we shall miss many interesting things which lie off the beaten track.

As for details, I pass them over because of lack of time. But mistakes are interesting. I plead guilty personally to an incurable tendency to make mistakes —trivial, annoying mistakes. In fact, on working over the first problem (as I thought I should before talking about it here), I first got imaginary frequencies! It was one of those ridiculous slips, omitting a factor 2 through carelessness. I wish I knew how to avoid doing this sort of thing. There are those who hold that a mistake in principle is a very serious sin, whereas a mistake in detail is easily forgiven. Frankly, I do not know the proper philosophy here, and would welcome advice.

I have not had time to say much about the third problem. I put it in as a foil for the other two, regarding it as a problem which lacked the practical appeal of the first and the aesthetic appeal of the second. Possibly I have not done justice to it in so treating it.

THE NEGLECTED SYNTHETIC APPROACH

L. M. Kelly, University of Missouri

The writer would like to enter a mild protest against the existing order of things in the field of mathematical education. This is, of course, not an unusual

desire. These remarks are directed wholly toward the training offered a mathematics major. Our principal thesis is that the synthetic approach, which was once so popular, has been relegated to a role of unwarranted obscurity. The classical conflict of the 1800's between the proponents of the analytic and synthetic methods has happily long since been dead and buried and it is not our intention of making any effort to revive it. However the fact that the analytic method is admittedly the most potent tool at the disposal of a mathematician does not warrant the complete disregard of all the other equipment. Long usage allows us to employ the terms analytic and synthetic without very precise definition. It is not altogether certain however that even rough agreement as to their meaning is generally attained. For purposes of this discussion no sharp delineations need be made but we will point out that our usage does not confine the reference to geometry nor does it indicate the absence or presence of coördinates as is sometimes done. Possibly a rough synonym for our use of the term synthetic could be "unorthodox." To the writer the absence or presence of "machinery" is the criterion but this probably does little to clarify the issue.

An example which illustrates the points we have in mind is the well known problem "*to find a point the sum of whose distances from three fixed points is a minimum.*" There are any number of published solutions to this problem ranging from methods of high school geometry (Johnson's *Modern Geometry*) to relatively sophisticated methods employing Lagrange multipliers. The analytic attack is obvious. A coördinate system is introduced, an expression for the distance sum is obtained, this expression is then differentiated partially, first with respect to x and then y, the two results are set equal to zero and the resulting equations solved simultaneously for the coördinates of the desired point. This sounds simple but, as a matter of fact, the algebra is quite involved. (See Goursat-Hedrick, vol. 1, page 130.) Even if the equations could be conveniently solved the result would be the coördinates of the point, which are not of particular interest.

The interesting property of the point is revealed by the following simple and quite natural argument which is essentially that contained in *What is Mathematics* by Courant and Robbins. Let A, B, and C be the three fixed points and P the minimizing point. (We assume its existence.) Consider the ellipse defined by the two foci A and B and passing through the point P. We maintain that the segment CP is normal to this ellipse, otherwise the foot P' of the normal to the ellipse from C would provide us with a point the sum of whose distances from the three fixed points was less than the alleged minimum. It follows that the angles CPA and CPB are equal and similar considerations show CPB equals APB. Thus, in general, the three segments PA, PB, and PC make angles of 120 degrees with one another. Certain special cases require examination but it is not our purpose to carry out an exhaustive analysis. Certainly it will be conceded that this is a synthetic approach and that it is simple and quite natural as well as elementary. If we ask where in the course of a normal training, undergraduate or graduate, a student would be encouraged to pursue such a line of attack the answer would seem to be "nowhere." Such proofs would hardly be tolerated, let

alone encouraged, in the ordinary calculus course. Similarly in all the other traditional courses with the possible exception of one in College Geometry. It seems regrettable that more institutions do not offer a course or two of the *pot pourri* variety built around a book like *What is Mathematics* where the synthetic method could be exploited along with the analytic.

A further advantage in addition to those already mentioned might be pointed out. Very often the synthetic approach makes the proof of an interesting proposition accessible to a larger group than would be the case if more sophisticated methods were employed. Such is the case in the following problem of obvious interest to cartographers and navigators. Problem: *Is it possible to map a spherical cap congruently (i.e., isometrically) onto the plane.* If a navigator tried to find out from a mathematician he would probably be confronted with some of the technical jargon of differential geometry such as Gauss curvature and differential forms. A very simple synthetic argument however is available. We first appeal to a lemma already proved in this MONTHLY (May, 1947, p. 283) and requiring nothing more advanced for its proof than some spherical trigonometry. This states that if a spherical triangle be congruently reproduced in the plane (that is with length of sides preserved) each angle of the plane triangle is less than the corresponding angle of the spherical triangle. This granted, we imagine, if possible, a one to one distance preserving map of a spherical cap onto the plane and ask what this mapping would do to the points A, B, and C forming an equilateral spherical triangle and its circumcenter D. A, B, C would correspond to the vertices A', B', and C' of a plane equilateral triangle and D would thus have to correspond to D' its circumcenter. But this is impossible since then the triple ABD would be congruent to the planar triple $A'B'D'$ and both the angle D and D' would be 120 degrees contrary to the lemma. This proof is due to L. M. Blumenthal. It is sometimes claimed that such methods are only for the ingenious. There will continue to be a dwindling group of ingenious people if ingenuity is not encouraged and cultivated.

A PROBLEM OF COLLINEAR POINTS

H. S. M. COXETER, University of Toronto

I would like to begin by expressing my wholehearted agreement with Mr. Kelly's plea for the synthetic approach. But my definition of "synthetic" would differ slightly from his. Synthetic geometry is ultimately based on certain primitive concepts and axioms, appropriate to the particular kind of geometry under consideration (e.g., projective or affine, real, complex or finite). Each problem belongs to one kind (or to a few kinds), and I would call a solution synthetic if it remains in that kind, analytic if it goes outside. The use of coördinates is one way of going outside; the use of trigonometry is another. These remarks are illustrated by the following problem, which was proposed by Sylvester in the *Educational Times*, Mathematical Question 11851, vol. 59 (1893), p. 98:

Let n given points have the property that the line joining any two of them passes

through a third point of the set. Must the n points all lie on one line?

Sylvester himself was doubtless aware of the negative answer in the case of the complex projective plane, where the nine points given by

$$x^3 + y^3 + z^3 - xyz = 0$$

(which are the points of inflexion of any one of the cubic curves $x^3+y^3+z^3 = cxyz$) lie by threes on twelve lines in such a way that every two of the points belong to such a set of three. Some years later, Veblen and Bussey described the finite affine geometries, one of which, called $EG(2, 3)$, consists of the nine points (x, y) whose coördinates are residues modulo 3, and the twelve lines

$$Xx + Yy + Z \equiv 0 \pmod 3$$

which join them. Since any two points determine a line, and every line contains three points, all the points in this geometry form a set of the desired kind (with $n=9$).

But the question as applied to the ordinary real plane remained unanswered for forty years, and began to seem as intractable as the four-color map problem. Finally, Grünwald established the affirmative answer: such points must all be collinear. Robert Steinberg transformed Grünwald's affine proof into a projective one (this MONTHLY, vol. 51 (1944), p. 169), and L. M. Kelly discovered a still shorter (and quite unrelated) Euclidean proof (see below).

It seems to me that parallelism and distance are essentially foreign to this problem, which is concerned only with incidence and order. Thus I would not regard the proofs by Grünwald and Kelly as strictly synthetic. Steinberg's projective proof becomes even more elementary when we transform it into a "descriptive" proof, using the single primitive entity *point* and the single primitive relation *between*, in terms of which lines and serial order can be defined (see Veblen, *Transactions of the American Mathematical Society*, vol. 5 (1904), pp. 353–371).

If the n points are not all collinear, some three of them must form a triangle ABC. Lines joining A to all other points of the set meet the line BC in B and C and possibly other points. Let P' be a point of BC not among these. Then the line AP' contains no point of the set except A. Joins of pairs of points belonging to the set meet AP' in A, P', and possibly other points. Let P be the *first* one of these, going from A towards P'. (P may possibly coincide with P'.) Then no join of two points of the set can pass through a point between A and P. We shall obtain a contradiction by showing that this "empty" segment AP is not really empty after all.

By its definition, the point P lies on a line containing at least three points of the set, say Q, R, S, named so that the segment PR contains Q but not S. (We do not care whether S lies beyond R or beyond P.) Since A and R belong to the set, the line AR contains a third point of the set, say O. Two cases now arise. If O lies between A and R, the line SO must intersect the segment AP (by Pasch's Theorem). Similarly, if O lies outside the segment AR, the line QO must inter-

sect the segment AP. In either case we have the desired contradiction. Hence the triangle ABC cannot exist, and the n points must in fact be collinear.

Alternative proof of collinearity, by *L. M. Kelly*. Assume the contrary. If the points be labeled $p_1, p_2, \cdots, p_n$, then there exists a point p_i and a line p_jp_k such that the perpendicular distance from p_i to the line p_jp_k is the shortest non-zero perpendicular distance from any of the points to any of the lines defined by pairs of the points. Let q be the foot of the perpendicular from p_i to the line p_jp_k. On the line p_jp_k there is at least one more point of the set, p_l. At least two of p_j, p_k, p_l must lie on one side of q. Suppose for definiteness that p_jp_kq is the order (p_k may coincide with q). Then the distance from p_k to the line p_ip_i provides us with a contradiction since it is shorter than the assumed minimum p_iq.

CAN WE TEACH GOOD MATHEMATICS TO UNDERGRADUATES?

R. G. Helsel and T. Radó, Ohio State University

We decided upon our topic shortly after Professor Birkhoff invited us to contribute to this symposium. Later Professor Pólya sent us detailed information concerning the nature of the symposium and we realized then that our topic was perhaps too general; however, we felt the question is of such import to teachers of college mathematics that it should not be put aside. One of the goals of our instruction certainly should be to show the student mathematics of the highest caliber, if we can.

Let us consider then the ingredients of good mathematics. First, it must be *relevant*. Fortunately, most of the subjects we teach are filled with pertinent topics. Calculus, for example, is overflowing with relevancy. Second, good mathematics must be *rigorous*. In other words, all of the reasons must be given. Some persons feel that to teach calculus rigorously the student must learn his ϵ's and δ's and be acquainted with the real number system as defined by Cantor or Dedekind. This is certainly not the case. In fact the sophomore regards any discussion of the real number system as being irrelevant, and it is, at that level. Finally, good mathematics must be *elegant*. If we fail to show our students elegant mathematics, we deprive them of the very thing which affords us our greatest pleasure.

Having listed the ingredients of good mathematics, there remains the question of whether it is possible to serve such a dish to the undergraduate. To answer this, consider the three parts of a theorem: *hypotheses*, *proof*, and *conclusion*. The conclusion, which we assume to be interesting and relevant, is fixed. There may be leeway in the method of proof, but the real freedom is in the choice of assumptions. Thus we propose to *pack the hypotheses* with carefully chosen assumptions that have the power to push through the proof in a rigorous and striking manner and yet are either obvious to the student or acceptable on the basis of past experience. Existential and qualitative assumptions are more likely to meet these requirements than quantitative assumptions.

We shall illustrate this general principle of *packing the hypotheses* by an ex-

ample from calculus. Perhaps our example is not well chosen, but it has the advantage of being brief. Anyone who is interested can find better examples for himself. Let us establish then the formula $d(\log_a x)/dx = c/x$. This certainly is a relevant and interesting relation for the student, since he already knows that $d(x^{n+1})/dx = (n+1)x^n$ if $n \neq -1$. The student also knows, or believes, that the graph of $\log_a x$ is continuous, monotone increasing, and has a tangent at each point. We shall use as our first hypothesis: H_1. *$d(\log_a x)/dx$ exists for $x>0$, call the derivative function $g(x)$.* As our second hypothesis we shall use the one functional relation involving logarithms which the sophomore does know: H_2. $\log_a x + \log_a \alpha = \log_a \alpha x$ *for* $x>0$ *and* $\alpha>0$. Differentiating the left side of this equation, regarding α as a constant, we obtain, through H_1, $g(x)$. Differentiating the right side, with the help of the chain rule for the derivative of a function of a function, we obtain $\alpha g(\alpha x)$. Now let x have the value 1 to obtain the equation $g(1) = \alpha g(\alpha)$, or $g(\alpha) = g(1)/\alpha$, and the proof is complete. Note that the value of the proportionality constant is the slope of the graph of $\log_a x$ at the point where it crosses the x axis, an interesting fact which is often overlooked. Similar proofs can be made for the derivative formulas of the trigonometric and exponential functions.

Thus it is our conviction that by the device of *packing the hypotheses* it is possible to show our students mathematics of the highest caliber. Of course the teacher must use great care in selecting relevant topics and creating interest in them. Also he must choose existential and qualitative hypotheses which are completely acceptable to the student. If the assumptions are properly chosen, the teacher can follow through with a rigorous and striking proof which will show the student what mathematics is really like. In addition the student will soon become aware of the power of existential assumptions and, it is to be hoped that, those who go on to advanced courses will have some interest in existence proofs.

Finally, we might ask whether it is following a high or low path to *pack the hypotheses* as suggested? In reply, consider how the working mathematician grabs everything that is plausible and relevant when he sets about to verify a surmise. Afterwards he may check through the details of theorems that he uses, but initially he merely assumes that they are valid. For example, most mathematicians have used the Jordan Curve Theorem at some time or other, but few indeed have ever gone through a rigorous proof. So it appears that the principle of *packing the hypotheses* is not only a means to show our students good mathematics but it is also in keeping with the manner in which mathematicians actually do their work.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 798. *Proposed by J. M. Elkin, Chicago, Ill.*

Prove that $\Pi(n_i!)$, where Σn_i is constant, is a minimum when $\Sigma|n_i - n_j|$ is a minimum, and that, consequently, the most likely distribution of the four suits in a bridge hand is four cards of one suit and three cards each of the other three suits.

E 799. *Proposed by Leo Moser, University of Manitoba*

Prove that for all n

$$\sum_{r=0}^{n}\left[2^{n-r}\binom{n}{r}^2 - \binom{2n-r}{n}\binom{n}{r}\right] = 0.$$

E 800. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The polar planes, with respect to a tetrahedron, of the isotomic conjugates of a set of collinear points are coaxal.

Note. If the cevians AP, BP, CP, DP, for a tetrahedron $ABCD$, of a point P, cut the planes of the faces BCD, CDA, DAB, ABC, in A', B', C', D', then the isotomic conjugate P' of P is the point of concurrency of the lines joining A, B, C, D to the isotomic conjugates A'', B'', C'', D'' of A', B', C', D' in the faces of the tetrahedron.

SOLUTIONS

Professor Umbugio's Simple Scheme

E 766 [1947, 223]. *Proposed by H.E.G.P.*

Professor Umbugio, who was introduced to our readers in our foregoing April number, invented a remarkable scheme for reviewing books. He divides the time he allows himself for reviewing into three fractions, α, β, and γ. He devotes the fraction α of his time to a deep study of the title page and the jacket. He devotes the fraction β to a spirited search for his name and for quotations from his works. Finally, he spends the fraction γ of his alloted time in a proportionally penetrating perusal of the remaining text. Knowing his characteristic taste for simple and direct methods, we cannot fail to be duly impressed by the differential equations on which he bases his scheme:

(1) $$dx/dt = y - z, \qquad dy/dt = z - x, \qquad dz/dt = x - y.$$

He considers a system of solutions x, y, z which is determined by initial conditions depending on a (small) parameter ϵ, independent of t. Therefore x, y, and z depend on both t and ϵ, and we appropriately use the notation:

(2) $$x = f(t, \epsilon), \qquad y = g(t, \epsilon), \qquad z = h(t, \epsilon).$$

The functions (2) satisfy the equations (1) and the initial conditions

(3) $$f(0, \epsilon) = 1/3 - \epsilon, \qquad g(0, \epsilon) = 1/3, \qquad h(0, \epsilon) = 1/3 + \epsilon.$$

Professor Umbugio defines his important fractions α, β, and γ by

$$\lim_{\epsilon \to 0} f(2, \epsilon) = \alpha, \qquad \lim_{\epsilon \to 0} g(5, \epsilon) = \beta, \qquad \lim_{\epsilon \to 0} h(279, \epsilon) = \gamma.$$

Deflate the Professor! Find α, β, γ without much numerical computation.

I. *Solution by R. E. Greenwood, University of Texas.* From Professor Umbugio's equations (1) we readily find that

$$dx/dt + dy/dt + dz/dt = 0,$$
$$x(dx/dt) + y(dy/dt) + z(dz/dt) = 0.$$

Integrating and taking into account the initial conditions (2), we obtain

$$x + y + z = 1,$$
$$x^2 + y^2 + z^2 = 1/3 + 2\epsilon^2,$$

whence we deduce that

$$(x - 1/3)^2 + (y - 1/3)^2 + (z - 1/3)^2 = 2\epsilon^2.$$

Thus, as $\epsilon \to 0$, we see that x, y, z each approaches $1/3$, no matter what t may be, and hence

$$\alpha = \beta = \gamma = 1/3.$$

Geometrically, the solutions of (1) are concentric circles on the plane $x+y+z=1$, with center $(1/3, 1/3, 1/3)$ and radii $\epsilon\sqrt{2}$. When $\epsilon \to 0$ we obtain the point $(1/3, 1/3, 1/3)$.

II. *Solution by C. F. Pinzka, North Plainfield, N. J.* Solution of (1) by the conventional methods yields

$$x = a_1 + a_2 \cos \sqrt{3}\, t + a_3 \sin \sqrt{3}\, t,$$
$$y = b_1 + b_2 \cos \sqrt{3}\, t + b_3 \sin \sqrt{3}\, t,$$
$$z = c_1 + c_2 \cos \sqrt{3}\, t + c_3 \sin \sqrt{3}\, t,$$

where the a's, b's and c's are constants which satisfy (1) and (3). Substituting in (1) and (3) and solving for the constants gives

$$x = 1/3 - \epsilon \cos \sqrt{3}\, t - (\epsilon \sin \sqrt{3}\, t)/\sqrt{3},$$

$$y = 1/3 + (2\epsilon \sin \sqrt{3}\, t)/\sqrt{3},$$
$$z = 1/3 + \epsilon \cos \sqrt{3}\, t - (\epsilon \sin \sqrt{3}\, t)/\sqrt{3}.$$

Taking the required limits, we find that $\alpha=\beta=\gamma=1/3$.

Also solved by Paul Brock, G. Y. Cherlin, E. S. Keeping, and W. D. Lambert.

An Application of Pappus' Theorem

E 767 [1947, 223]. *Proposed by Milton Schwartz, Temple University*

From any point P on BC of parallelogram $ABCD$ line segments are drawn to A and D. From any point Q on AD line segments are drawn to B and C. Through the intersections of these four segments (PA, PD, QB, QC) a line is drawn meeting AB in R and CD in S. Prove that BR equals DS.

Solution by Paul Brock, Hunter College. Let PA, QB intersect in T and PD, QC in V. Then, by Pappus' theorem, T, V, and the intersection, L, of the diagonals are collinear. It is obvious that triangles LRB and LSD are congruent. Therefore BR equals SD.

Also solved by W. G. Brady, D. H. Browne, G. Y. Cherlin, H. S. M. Coxeter, W. P. DeWitt, William Douglas, H. E. Fettis, L. M. Kelly, B. R. Leeds, D. K. Pease, Kaidy Tan, Maud Willey, and the proposer.

Editorial Note. This problem has been proposed in the problem department of *School Science and Mathematics* and solutions have appeared in the Nov. 1931 and the Oct. 1932 issues. Also, Problem E 129 of this MONTHLY (May, 1935) is the same problem. In these references one can find solutions using only high school methods.

As a slight generalization we may take $ABCD$ to be a trapezoid, with AB and DC as the parallel sides. Then we may show that $AR/RB = CS/SD$.

Sums of Consecutive Integers

E 768 [1947, 224]. *Proposed by Irving Kaplansky, University of Chicago*

A number n has the property that for any $p<q<n$,

$$S = p + (p+1) + \cdots + q$$

is never divisible by n. Show that this is true if and only if n is a power of 2.

Solutions by S. H. Gould, University of Wisconsin. Set $n=m\cdot 2^k$, with m odd. If $m=1$, then $2S=(q+p)(q-p+1)$, which has one factor even and the other odd, cannot be divisible by $2n=2^{k+1}$, since its even factor is less than $2n$. But if $m>1$, then S is divisible by n, with $0<p<q<n$, if we take

$$q = (m + 2^{k+1} - 1)/2,$$

and

$$p = (1 + q - 2^{k+1}) \qquad \text{for } m > 2^{k+1},$$

or

$$p = (1 + q - m) \qquad \text{for } m < 2^{k+1}.$$

As an interesting extension of this problem we can establish the following theorem.

Let $f(n)$ denote the number of ways in which p and q can be chosen with $0<p<q<n$ such that

$$S(p, q) = p + (p + 1) + \cdots + q$$

is divisible by n. Then

$$f(n) = \sum_{k} [(k, n)(2n - k - 1)/2n],$$

the sum being taken for all odd numbers k not greater than $2n-3$ and not prime to n. The symbol (k, n) denotes the H. C. F. of k and n, and $[a/b]$ the greatest integer in a/b.

For all choices of p and q without the restriction $q<n$, each value of $S(p, q)$ will be given exactly once by the elements of the infinite square matrix $a_{ij}=i(2j+1)$, since

$$S(p, q) = a_{ij}$$

with

$$i = (q + p)/2, \qquad j = (q - p)/2, \qquad \text{for } p + q \text{ even}$$

and

$$i = (q + p - 1)/2, \qquad j = (q + p + 1)/2, \qquad \text{for } p + q \text{ odd.}$$

The restriction $q=i+j<n$ means that for fixed j we consider the multiples ik of the odd number $k=2j+1$ only for $i \leqq n-j-1=(2n-k-1)/2$. These will be multiples of n if and only if i is a multiple of $n/(k, n)$, (impossible if k is prime to n), that is, for $i=mn/(k, n)$ with

$$m = 1, 2, 3, \cdots, [(k, n)(2n - k - 1)/2n].$$

Also solved by Murray Barbour, Louis Berkofsky, Paul Brock, D. H. Browne, N. J. Fine, J. B. Kelly, L. M. Kelly, V. L. Klee, Jr., N. S. Mendelsohn, Leo Moser, G. C. Webber, and the proposer.

After proposing this problem the proposer located a partial statement of it in ex. 5, p. 83 of *Elementary Number Theory* by Uspensky and Heaslet.

Leo Moser stated and proved the following associated theorem:

If $a_1, a_2, \cdots, a_n$ are any n positive integers, then for some p and q, $p \leqq q \leqq n$, $S=a_p+a_{p+1}+\cdots+a_q$ is divisible by n.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4277. *Proposed by C. D. Olds, San Jose State College, California*

In a non-associative algebra, it is necessary to distinguish the possible interpretations of x^n. Thus, for example, in a non-commutative non-associative algebra x^3 can mean $x\cdot x^2$ or $x^2\cdot x$. In a general non-commutative non-associative algebra the number of interpretations of x^n is $2(2n-3)!/n!(n-2)!$. Is there a formula for the number of interpretations of x^n in a general commutative non-associative algebra?

4278. *Proposed by Peter Ungár, Budapest, Hungary*

Construct two divergent series, $\sum a_k$ and $\sum b_k$ with $a_1 \geqq a_2 \geqq \cdots \geqq 0$, $b_1 \geqq b_2 \geqq \cdots \geqq 0$, but such that if $c_k \leqq \min(a_k, b_k)$, $c_k > 0$, then $\sum c_k$ is convergent.

4279. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Prove that a tetrahedron $ABCD$, whose vertices A, B, C are the inverse, with respect to a sphere with center D, of a right-angled triangle, and the tangential tetrahedron $A_1B_1C_1D_1$ of that tetrahedron are each inscribed in the other and that the midpoints of AD_1, BC_1, CB_1, DA_1 are coplanar.

SOLUTIONS

Hermite Polynomials

4215 [1946, 470]. *Proposed by Hüseyin Demir, Columbia University*

Prove that the Hermite polynomials defined as follows

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$$

have the property

$$n! \sum_{p=0}^{n} \frac{H_p^2(x)}{p!} = H_{n+1}^2(x) - H_n(x) H_{n+2}(x).$$

Solution by Hsien-yü Hsü Yenching University, Peiping, China. In Polya-Szegö, *Aufgaben und Lehrsätze* II, pp. 294–295, Hermite polynomials are de-

fined as follows

$$h_n(x) = \frac{1}{n!}\, e^{x^2/2}\, \frac{d^n}{dx^n}\, e^{-x^2/2},$$

and satisfy the difference equation

$$nh_n(x) = -\, xh_{n-1}(x) - h_{n-2}(x), \qquad n = 2, 3, \cdots.$$

We notice that

$$H_n \equiv H_n(x) = (-1)^n n! h_n(x),$$

whence the difference equation is

$$H_n = xH_{n-1} - (n-1)H_{n-2}, \qquad n = 2, 3, \cdots. \tag{1}$$

Upon eliminating x from this equation and the analogous equations for H_{n+1} and H_{n+2} we obtain immediately

$$\begin{aligned} H_{n+1}^2 - H_nH_{n+2} &= n(H_n^2 - H_{n+1}H_{n-1}) + H_n^2 \\ &= n(n-1)(H_{n-1}^2 - H_nH_{n-2}) + nH_{n-1}^2 + H_n^2 \\ &= \cdots \\ &= n!\left\{(H_1^2 - H_2H_0) + \frac{1}{1!}\, H_1^2 + \frac{1}{2!}\, H_2^2 + \cdots + \frac{1}{n!}\, H_n^2\right\} \\ &= n! \sum_{p=0}^{n} H_p^2/p!, \end{aligned}$$

since $H_1^2 - H_2H_0 = 1$.

Solved also by F. E. Cothran, A. B. Farnell, L. M. Kelly, Norman Miller, S. T. Parker, W. A. Pierce, W. H. Spragens, M. S. Webster, M. Wyman, Professor Otto Szász's class, and the Proposer.

Editorial Note. Several solvers mentioned that equation (1) of the above solution is found in Dunham Jackson, *Fourier Series and Orthogonal Polynomials*, p. 176, ff. The solution by members of Professor Szász's class in Orthogonal Developments proceeds from the (so-called) Christoffel's formula

$$\sum_{p=0}^{n} \frac{H_p(x)H_p(y)}{p!} = \frac{H_{n+1}(x)H_n(y) - H_n(x)H_{n+1}(y)}{n!(x-y)}.$$

See Szegö, *Orthogonal Polynomials*, p. 102. Webster's solution employs the relation

$$H_m(x)H_n(x) = n! \sum_{r=0}^{n} \binom{m}{n-r} \frac{H_{m-n+2r}}{r!} \qquad (m \geqq n),$$

established by E. Feldheim, *Journal of the London Mathematical Society*, 1938, pp. 22–29.

Cesàro Continuity

4216 [1946, 470]. *Proposed by Herbert Robbins, Annapolis, Md.*

Write $\{x_n\} \overset{.}{\to} \alpha$ if $\lim_{n\to\infty} (x_1 + \cdots + x_n)/n = \alpha$. A function $f(x)$ is said to be C.c. (Cesàro continuous) at $x=\alpha$ if $\{x_n\} \overset{.}{\to} \alpha$ implies $f(x_n) \overset{.}{\to} f(\alpha)$. Show that if $f(x)$ is of the form $Ax+B$ then it is C.c. at every value of x, and that if $f(x)$ is C.c. at even a single value $x=\alpha$, then $f(x)$ is of the form $Ax+B$.

Solution by R. C. Buck, Brown University. The first part of the problem is trivial. For the second part suppose that $f(x)$ is C.c. at a single value α. By changing to parallel coördinate axes, we can take $\alpha=0$ and $f(\alpha)=0$. Then the sequence $\{a, b, c, a, b, c, \cdots\}$ is Cesàro convergent to zero if $a+b+c=0$. From the Cesàro continuity of $f(x)$ at zero we conclude that $f(a)+f(b)+f(c)=0$, or $f(a)+f(b)=-f(-a-b)$. Thus also $f(a)=-f(-a)$, whence for any x and y

$$f(x+y) = f(x) + f(y), \qquad f(nx) = nf(x).$$

If a sequence $\{x_n\}$ is Cesàro convergent to zero, so that $\lim \sigma_n = 0$ where $n\sigma_n = x_1 + \cdots + x_n$, then

$$\lim \frac{1}{n} \sum_{k=1}^{n} f(x_k) = \lim f(\sigma_n) = 0.$$

Hence $f(x)$ is continuous in the usual sense at zero, and by the additivity of $f(x)$, at all other points x. It is well known that the only continuous function obeying $f(x+y)=f(x)+f(y)$ is $f(x)=Ax$.

Solved also by P. T. Bateman, Colin Blyth, Jr., Harley Flanders, William Gustin, and J. B. Kelly.

Integers

4217 [1946, 471]. *Proposed by P. A. Piza, San Juan, P. R.*

Let a and b be positive integers whose sum is the square c^2 and whose difference is the cube d^3. Given that each of c^2, d^3 and a is a 4-digit integer and that the sum of the 4 digits included in each is equal to b, find a and b.

Solution by Monte Dernham, San Francisco. If

$$a = 10^3m + 10^2h + 10t + u, \qquad b = m + h + t + u,$$

then $d^3 = 999m + 99h + 9t$. Now the only four-digit cubes divisible by 9 are $12^3 = 1728$, $15^3 = 3375$, $18^3 = 5832$, $21^3 = 9261$. In any case $b = 18$ so that $c^2 = a+18 = d^3+36$. Now d^3+36 is a square for only the first of the cubes listed. Hence the unique answer is given by

$$b = 18, \qquad a = 1746, \qquad c^2 = 1764, \qquad d^3 = 1728.$$

Solved also by M. Aissen, Murray Barbour, R. G. Blake, Paul Brock, D. H. Browne, R. B. Herrera, N. Kaufman and R. J. Koch, Helen Nickerson, Clay Perry, C. F. Pinzka, W. A. Pierce, E. D. Schell, Maud Willey, and the Proposer.

Integers with k Factors

4220. [1946, 537]. *Proposed by P. Erdös, Syracuse University.*

Let m be a positive integer with no prime factors greater than n and $m \leqq n^{(k+1)/2}$. Then m can be written as the product of k integers $\leqq n$; the exponent $(k+1)/2$ is the best possible.

Solution by N. J. Fine, Washington, D. C. Let m satisfy the given conditions' and suppose that m cannot be written as the product of k integers $\leqq n$. Write m as a product of primes:

$$m = \prod_{i=1}^{r} p_i, \qquad p_1 \leqq p_2 \leqq \cdots \leqq p_r.$$

We can define $A_1, A_2, \cdots, A_k$, and $r_1, r_2 \cdots, r_k$ by

$$A_1 = \prod_{i=1}^{r_1-1} p_i \leqq n, \qquad p_{r_1}A_1 > n,$$

$$A_2 = \prod_{i=r_1}^{r_2-1} p_i \leqq n, \qquad p_{r_2}A_2 > n,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$A_k = \prod_{i=r_{k-1}}^{r_k-1} p_i \leqq n, \qquad p_{r_k}A_k > n.$$

The possibility of continuing this construction at least as far as A_k is guaranteed by our assumptions. Clearly $A_{j+1} \geqq P_{r_j}$, $j = 1, 2, \cdots, k-1$, and $m \geqq A_1A_2 \cdots A_kp_{r_k}$. Hence

$$\begin{aligned} m^2 &\geqq (A_1A_2 \cdots A_kp_{r_k})^2 = (A_1A_2)(A_2A_3) \cdots (A_{k-1}A_k)(A_1p_{r_k})(A_kp_{r_k}) \\ &\geqq (A_1p_{r_1})^2(A_2p_{r_2}) \cdots (A_{k-1}p_{r_{k-1}})(A_kp_{r_k}) > n^{k+1}. \end{aligned}$$

But this is a contradiction and proves the first part of the theorem.

Given a fixed positive integer k and an arbitrary $\epsilon > 0$, let

$$\delta = \min\left(\frac{1}{2}, \frac{\epsilon}{k+1}\right).$$

Then there is an integer $n_0(\delta)$ so large that for all $n > n_0(\delta)$, there exists a prime p satisfying

$$n^{1/2} < p < n^{1/2+\delta}$$

The integer $m = p^{k+1}$ has no prime factors exceeding n, cannot be written as the product of k integers $\leqq n$, and is less than $n^{(k+1)/2+\epsilon}$. Thus the proposed exponent is the best possible.

Solved also by P. T. Bateman.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, 27, N. Y. and not to any of the other editors or officers of the Association.

Introduction to the Theory of Equations. By Lois W. Griffiths. New York, John Wiley and Sons, Inc., 1947. 9+278 pages. $3.50.

The problem of content in a course in the Theory of Equations is one that is open to discussion. Many of the topics considered are, of course, standard with all instructors. However, the majority of the books in this subject offer more material, albeit briefly, than can possibly be presented in the usual half year allotted to this course. Therefore, it is not only possible, but necessary, that each instructor use some selection in outlining his course. On the other hand, Miss Griffiths has chosen to present in her book somewhat less than the usual number of topics, but to present the material selected in greater detail. While the additional detail is commendable, the merit of this book in the eyes of the individual instructor will probably be determined to a large degree by his or her agreement or lack of agreement in the choice of subject matter.

As the author has stated in the preface, "Development from the particular to the general is an outstanding feature of this book." In each case, she has gone through the details of a proof with a particular example before attempting to prove the general case. For instance, she isolates the real roots of a particular quartic equation in complete detail before embarking upon a general discussion of Sturm's Theorem. Using this approach, the major proofs are presented at least twice. This feature makes this book particularly useful for the student who proposes to master the material with no aid other than the text.

Before listing in more detail the topics discussed, it is well to note that only the algebraic equation is considered. This is particularly worthy of note since Newton's method for the approximation of the real roots of an equation is not mentioned, nor is the method of Budan for the isolation of the real roots of an equation mentioned at any point in the book. The first chapter is devoted to the binomial equation, roots of unity, and a very elementary treatise on complex numbers. The second chapter includes the solution of the cubic and quartic equation, with a discussion of the discriminant of each. The third and fourth chapters give some general theorems on the roots of an equation (including the factor theorem, remainder theorem, and upper and lower bounds on the real roots), Sturm's theorem, and Horner's method for the computation of real roots. Chapters five and six are used to develop the theory of determinants, with an introduction of the concept of a matrix. Systems of linear equations are discussed in chapter seven. Chapter eight gives a more rigorous, though brief, treatment of complex numbers followed by a statement of the fundamental theorem of algebra. The last of the nine chapters is devoted to a short discussion of symmetric functions. There are many good problems throughout the

book, and the answers are given for all of the odd-numbered problems.

This book presents a somewhat different menu than many texts in this field. In general the selection presented here would emphasize the initial word in the title, and the presentation is sufficiently elementary that no student who would ordinarily be permitted to enroll for this course should have any difficulty in comprehending the material set forth. To this extent, this book is to be recommended. There will be some, however, who will question whether or not this book may have too little material beyond a good text in college algebra. As stated before, the answer will lie in the agreement or disagreement of the individual instructor with the material presented.

R. L. WILSON

Introduction to Mathematical Statistics. By P. G. Hoel. New York, John Wiley and Sons, Inc., 1947. 10+258 pages, $3.50.

The reviewer's opinion that the minimum of mathematical preparation for a satisfactory introductory course in mathematical statistics is a year of the calculus is widely shared but so far no really suitable textbook for such a course has been available. It is with considerable anticipation, therefore, that he noted that Hoel's new book is intended to fill this need, particularly in view of the author's known competence in this field. But a rapid inspection of the contents reminds one of the difficulties one faces in planning a first course in mathematical statistics. (1) Is this first course also to be in most cases the terminal course? If it can be generally admitted that any real statistical competence is not likely to be gained in a one-year course, then an author can more easily avoid spreading himself too thin. (2) Is one to concern himself chiefly with the mathematics of statistics which can be so formal as to be almost meaningless or is the science of statistics also to be a matter of principal interest? The reviewer has always found the formal mathematics in elementary statistics the easier half of what is to be taught; to give the student an orientation in a new discipline is more difficult. (3) The students electing such a course vary not only in their mathematical preparation and maturity but also even more widely in their previous acquaintance with the uses to which statistical methods may be put. The most enthusiastic students are commonly those who have already acquired a considerable practical acquaintance with the methods taught but who are now satisfying a most laudable desire to gain some real understanding of them. These and not average students who have just completed a calculus course are the ones to whom the more purely mathematical type of course should be taught. It appears to the reviewer that Hoel's book is best adapted to students with mathematical maturity beyond that usually conferred by a calculus course and with some previous acquaintance with statistical methods.

The first seven chapters cover what has heretofore commonly been the material for an introductory course but the mode of treatment is a distinct departure. Not many words are wasted, for the job is done in 127 pages. Little is actually omitted that ought to be included and there are frequent numerical

illustrations which indicate the importance and nature of the applications. What is said is almost all accurate and one can take little exception to the mathematics. But the discussion is always brief and the mathematics neat. Brevity with clarity is a merit of an elementary textbook provided it is taught by an instructor who can amplify the discussion and provide additional examples. But pedagogically speaking, mathematical elegance ought sometimes to be eschewed in favor of the direct attack. The obvious case in point here is the early introduction of the moment generating function and its free use. This gets in a concise and pleasing way many results, and the student of mathematical statistics should not go for long without making its acquaintance; but had not beginners better first calculate the variance of a normally distributed variable or of one obeying a binomial distribution law by direct integration or summation? Finally in these seven chapters actual computing methods are treated very briefly indeed in spite of the fact that there is much to be taught about computation and that no one becomes a competent statistician without having a great deal of computation behind him.

The remaining five chapters include an introduction to the small sample significance tests and to χ^2 tests which have now become standard. The exposition is still brief but it is still a long step beyond what has heretofore been available in an elementary text in accuracy and mathematical soundness. The reviewer does not see why at least the geometrical proof of the independence of the mean and variance in samples from normal and the development of the approximation which leads to the χ^2 test of goodness of fit could not have been attempted in view of other derivations that are included. The reviewer does have differences with the author with respect to the Student-Fisher t test. On page 143 it ought to be stated that this test does not remedy the defect of the large sample test. In fact for small samples the defect cannot be remedied; instead in strict accuracy we are forced to use a different test. Again in both numerical illustrations, on this page and on page 146, the regions of rejection are both taken as one-sided, which is only correct if it were decided in advance of performing the experiment that the alternate hypotheses admissible were also one-sided, which seems very debatable.

In addition some rather recent material is included. Linear discriminant functions and the theory of runs are introduced with developments that seem really substantial for college juniors. Non-parametric tests, interval estimation (confidence intervals), sampling inspection methods including sequential analysis, stratified sampling, some of the Neyman-Pearson theory of significance tests, and a little about maximum likelihood methods are all included briefly, some of them so very briefly in comparison with what would be required to give any real comprehension of them, that it makes their value to the student doubtful.

Very few typographical errors were noted. The book suffers from the very common inconvenience of requiring the reader to search for the page on which, say, (3), Chapter V referred to is to be found. At the end of each chapter refer-

ences are given to other, fuller, or more mathematical treatments of material in that chapter. Good lists of problems follow each chapter, and the instructor will find often included as exercises nontrivial derivations of theoretical results which supplement the text.

I think it is evident that the reviewer's chief concern is that Hoel's book may prove a little more mature and to treat too much material a little more concisely than is best for beginners fresh from a calculus course. A really experienced instructor could undoubtedly amplify and supplement the text and have in his students' hands a basic reference book that is much more sound, more complete, and more modern than anything that has been available at anywhere near the proper level. The book does represent a very long advance in the right direction.

C. C. CRAIG

A First Course in Mathematical Statistics. By C. E. Weatherburn. Cambridge, at the University Press; New York, The Macmillan Company, 1946. 15+271 pages. $3.50.

This book has a good many points of similarity to the even more recent text by Hoel, reviewed above. It is written at about the same mathematical level and for the most part, as is natural, deals with the topics found in the other book. It is, however, even more concerned with the mathematics of statistics, so far as its mathematical level permits, and by not including an account of some of the more recent developments dealt with in Hoel, space is found for a fuller treatment of some of its subjects. Greater mathematical clarity seems to be obtained at points, especially in the first half, but the discussion of the meaning of the results from the viewpoint of application to experimental work is even more brief. This is primarily a book for a comparatively mature student who already has a working knowledge of methods of statistical investigation, who wishes to gain a sounder understanding of their validity, and who has a good command of the calculus or a little more as his mathematical equipment. For students who are quite new to statistics, a good deal would need to be added by the instructor to keep a course based on this book from being quite formal.

The first five chapters deal with moments, probability, the binomial, the univariate and bivariate normal and the Poisson frequency functions and finally bivariate regression theory. Confined to 105 pages the discussion is necessarily often brief. Here also the moment generating function is introduced and freely used and Weatherburn goes beyond Hoel in deriving Sheppard's adjustments (as corrections good on the average) and in defining and using Thiele's semin-variants (or cumulants). The introduction to probability is not so very compressed as in Hoel's book and the student is likely to feel better prepared to go on to the study of frequency laws. As the author says, he has deliberately avoided the more modern mathematical formulations of the theory of probability, but the reviewer would be better pleased if it were at least mentioned that there are logical difficulties in the concept of the limit of an observed frequency ratio as

the number of observations increases indefinitely.

The remaining seven chapters are concerned with sampling theory, sampling distributions, and tests of significance. A rather clear division is made between results only valid for large samples and those good for samples of any size. The distributions of the Student-Fisher t and of the variance ratio in samples from normal and the χ^2 distribution are derived and the usual applications are made. Interesting features are the introduction of the Γ- and β-type distribution functions as the basic functions for the distributions used for the exact tests in samples from normal universes, and the derivation of the χ^2 distribution by the method used by Fisher in 1922. There is a chapter on the analysis of variance and covariance in which the exposition is clear for the simple two, three and four component (Latin square) analyses of variance but is less clear for the one and two component analyses of covariance dealt with. The final chapter on multivariate regression theory also seems to suffer from an effort to achieve brevity.

A great deal of what the reviewer said in a general way concerning Hoel's book could be repeated here. Again it is very pleasing to have another book on mathematical statistics at this level that one can recommend to students as mathematically sound and adequate. There are good lists of problems and some of them embody additional theoretical developments that could have been included in the text. Computing methods are not given a very prominent place.

There are a good many places where the reviewer would like to see fuller discussion and illustration; fewer where he would want to revise what is said. In the discussion of the standard errors of sample parameters there should be some mention of the efficiency of a statistic as well as its bias and the two concepts should be distinguished from each other. The development of the large sample standard error of a sample variance and the account of the approximation obtained is really not satisfactory. Presumably the author doesn't mean to say on page 179 that the sum of two stochastic variates could be independent of either summand. In the comparison of two sample means by the Student-Fisher t test, the assumed equality of the variances in the universes sampled needs emphasis and the fact that this assumption can be tested by the use of the variance ratio needs to be included and illustrated. In fact the student is not shown at all how to find the probability that the variance ratio be less than a given proper fraction. The relation between the variance ratio with one degree of freedom and the Student-Fisher t ought to be pointed out.

But these are minor criticisms. We really have here another very useful reference book and an excellent text for a properly prepared class. The lack of books where statistical mathematics could be found at this level has been a serious handicap in statistical instruction and this book and Hoel's will be of much aid to both students and teachers.

C. C. CRAIG

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

EDITORIAL NOTE: The request for *Club Reports* for 1946–47 has been responded to most satisfactorily by the advisers of mathematics clubs. The reports will appear on these pages in the order in which they have been received. If there are clubs who wish to report on the year's activities please send the material to the Editor of this department.

CLUB REPORTS, 1946–47

Mathematics Club, Purdue University

The Purdue *Mathematics Club* has completed a most successful first year. Following the organizational meeting, papers have been presented at each meeting. Some of the topics that were discussed include:

A trip through the universe, by Professor Carl Holton

Mathematics and bridge, by Professor I. W. Burr

Closed curves having a pencil of constant chords, by Professor C. K. Robbins.

One activity of the club consisted of a series of problems presented in the school paper. Solutions were submitted by interested persons on the campus. The correct solution was then published in a subsequent edition of the paper.

Officers for 1946–47: President, Arthur F. Sterling; Secretary, Nancy F. Van Ness; Treasurer, Nelson J. Herbert; Program Chairman, William F. Haldeman; Publicity Chairman, Charles O. Davis.

Kappa Mu Epsilon, Montclair State Teachers College

The papers presented at the regular monthly meetings were:

Navigation, by Lawrence Campbell, Walter Rissler and Robert Poppke.

The number e, by Professor Howard F. Fehr

Map making, by June Van Hoeven

Ruler and compass constructions, by Max Sobel.

Twenty new members were initiated during the year.

In April, a banquet was held for the double purpose of celebrating the twentieth anniversary at Montclair of faculty advisor Professor V. S. Mallory, and renewing friendships between alumni and undergraduate members.

The semester closed with the traditional party at Dr. Mallory's home. At this time awards were given to the three seniors who had attained the highest marks in mathematics throughout their four years at college. Mrs. Florence Elder, Max Sobel, and Bessie Kaplan each received a copy of Gaylord Merriman's *To Discover Mathematics*.

The officers who served during the year 1946–47 were: President, Dorothy McGrory; Vice-President, Muriel Roversi; Recording Secretary, Bessie Kaplan; Treasurer, Helga Schrank; Corresponding Secretary, Professor H. F. Fehr.

The newly elected officers for the year 1947–48 are: President, Walter Rissler;

Vice-President, Ann McCumsey; Recording Secretary, Ruth Muller; Treasurer, Jean McRae; Corresponding Secretary, Professor H. F. Fehr.

Mathematics Club, Illinois Institute of Technology

The policy of the club during the year was to present a number of lectures which would be of interest to the engineering students and beginning mathematics majors as well as presenting several lectures of an advanced nature for those with more mathematical background. This policy proved quite successful.

The lecturers, both student and guest speakers, presented the following papers:

Some remarkable theorems concerning moving figures, by Dr. L. R. Ford, who gave a very popular lecture dealing with the curve traced by a fixed point on a line segment of fixed length whose end points follow any two curves.

The probability function and the gamma function, by Robert Teasdale

Prime numbers, by Milton Searl

Motivation interaction, by Dr. Anatol Rapaport, whose lecture evoked considerable interest and discussion. The paper dealt with the mathematical evaluation of the amount of work a person of given psychological characteristics would do under varying circumstances.

Non-euclidean systems of geometry, by Malcolm Smith

Bessel functions, by Chandler Sammons

What is a curve? by Dr. Karl Menger, in which the main historical concepts as well as the present concepts of a curve were presented and proved quite fascinating to many of the club members. Menger's Universal Curve was also discussed.

Differential equations, by Don Mandel Friedlen

Vectors and fields, by H. Flanders.

In addition to their regular meetings the club participated in Junior Week, an open house activity. Featured in the clubs display was an exhibit of calculating machines and a brachristochrome. First prize was won by the club in one phase of the competition.

Officers for the September 1946 term were: President, Rober Teasdale; Publicity and Program Director, Milton Searl; Secretary, Chandler Sammons; Faculty Advisor, Dr. Anatol Rapaport.

Officers for the February 1947 term were: President, Malcolm Smith; Publicity and Program Director, Louis Joseph; Secretary, Milton Searl; Faculty Advisor, Dr. Anatol Rapaport.

Pi Mu Epsilon, University of Arizona

On April 7, 1941, the local mathematical group became the *Arizona Alpha* Chapter of the national organization, *Pi Mu Epsilon*. Since that time, annual elections have been held and a cup awarded for achievement in mathematics.

Two years ago the proposal was made and passed which put the local chapter under student management with a member of the faculty popularly elected to

act as advisor. Since that time it has been the purpose to hold at least four dinner meetings during each scholastic year, the banquets being followed by an hour lecture on some interesting aspect of mathematics and given by any student sufficiently advanced in mathematics. The initiation banquet scheduled as speaker a faculty member, either local or from another institution. The public was invited to all of these talks.

The papers presented this year were:

Application of mathematics to radio, by Robert Masching, a junior student in electrical engineering

The construction of magic squares, by F. E. Haupt, Instructor in the department of mathematics

Mathematics as related to other great fields of learning, by R. J. Hannelly, Dean of Phoenix Junior College.

Robert O. Wenzel received the 1946 *Pi Mu Epsilon* cup.

Excluding various business meetings, nomination committee meetings and so on, the season's activities were concluded with a student-faculty picnic.

This year's officers were: President, Donald B. Marsh, Jr.; Treasurer, Paul R. Vandiver; Secretary, Louisa F. Simons; Faculty Advisor, Dr. D. L. Webb.

The officers elected for next year are: President, John F. Pfeiffer; Treasurer, John Williams; Secretary, Harriet W. Darley; Faculty Advisor, J. F. Foster.

Mathematics Club, Camp Dodge Annex of Iowa State College

The *Mathematics Club* was organized in November, 1946, to stimulate interest in mathematics among the freshman students of the college Annex. The Club was sponsored by the *Iowa Alpha* Chapter of *Pi Mu Epsilon*. The programs at monthly meetings emphasized those phases of mathematics which were usually not found in textbooks. The program topics included:

Generalizations of trigonometry, by Professor H. P. Thielman

Games of chance, by Professor F. A. Brandner.

For the final meeting of the school year a motor trip was made to the Drake University observatory at Des Moines, where Professor B. E. Gillam of Drake University gave an excellent lecture on popular astronomy.

Refreshments were served at each meeting at the Recreation Hall of the Camp Dodge Annex. Quarterly dues were assessed to each of the seventy-five members to help defray this cost.

Much able assistance in organizing and planning of programs was given by Professor Fred Robertson, head of the mathematics department at the Camp Dodge Annex, and Ralph Robinson, treasurer of Pi Mu Epsilon at Iowa State.

Officers of the club were: President, Robert Triska; Vice-President, John Hart; Secretary, Kenneth Brown; Treasurer, James O. Loy; Faculty Advisor, Mr. R. L. Lambert.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

ACKNOWLEDGMENT

The editor of the MONTHLY wishes to make grateful acknowledgment of the services rendered by the following persons who have refereed papers: C. R. Adams, R. P. Agnew, C. B. Allendoerfer, H. M. Bacon, Reinhold Baer, L. M. Bauer, E. T. Bell, Garrett Birkhoff, L. M. Blumenthal, C. B. Boyer, J. B. Brandeberry, H. W. Brinkmann, O. E. Brown, J. W. Calkin, F. E. Carr, R. V. Churchill, C. J. Coe, A. H. Copeland, N. A. Court, H. S. M. Coxeter, Wayne Dancer, J. L. Doob, Arnold Dresden, Ben Dushnik, J. D. Elder, Howard Eves, Herbert Federer, W. D. Feller, L. R. Ford, Orrin Frink, R. E. Gaskell, F. C. Gentry, Lois W. Griffiths, V. G. Grove, Marshall Hall, P. R. Halmos, G. E. Hay, E. H. C. Hildebrandt, R. T. Hood, Ralph Hull, W. R. Hutcherson, S. A. Jennings, Fritz John, L. S. Johnston, B. W. Jones, Irving Kaplansky, H. T. Karnes, Edward Kasner, A. J. Kempner, F. L. Kiokemeister, V. L. Klee, M. S. Knebelman, C. F. Kossack, E. P. Lane, R. E. Langer, H. D. Larsen, D. H. Lehmer, L. H. Loomis, V. O. McBrien, N. H. McCoy, S. W. McCuskey, Saunders MacLane, M. L. MacQueen, H. B. Mann, W. T. Martin, Margaret Mauch, W. E. Milne, J. R. Musselman, Ivan Niven, E. P. Northrop, J. A. Nyswander, L. F. Ollmann, B. C. Patterson, William Prager, E. J. Purcell, Tibor Radó, G. Y. Rainich, J. F. Randolph, W. T. Reid, Samuel Selby, C. H. Sisam, F. W. Sparks, C. E. Springer, J. L. Synge, Otto Szasz, J. M. Thomas, J. W. Tukey, E. P. Vance, R. W. Wagner, H. S. Wall, John Williamson, F. L. Wren, R. C. Yates, C. H. Yeaton, Oscar Zariski.

SIXTH EDITION OF ARCHIBALD'S OUTLINE

A sixth edition of the well-known *Outline of the history of mathematics* by Raymond Clare Archibald is now in process of preparation. Professor Archibald would be glad to receive criticisms and suggestions for improvement of the fifth edition.

Although the primary purpose of the Slaught Memorial Papers is to make available *new* expositions of worth-while subjects, there is such a widespread interest in the *Outline* that it has been decided to make an exception in this case. Accordingly, the sixth edition will appear as a Slaught Memorial Paper, and all subscribers to the MONTHLY will receive a copy free of charge.

CORRECTION TO A RECENT REPORT

There are three errors in my recent report of the 1947 William Lowell Putnam Competition. They may be corrected by adding the name of Gerard Washnitzer of Brooklyn College to the list of contestants receiving honorable mention,

by adding the University of British Columbia to the list of institutions entering teams, and by removing Boston College from the latter list.—G. W. MACKEY.

INSTITUTE FOR ADVANCED STUDY

The School of Mathematics of the Institute for Advanced Study will allocate a small number of stipends to gifted young mathematicians and mathematical physicists to enable them to study and to do research work at Princeton during the academic year, 1948–1949. Candidates must have given evidence of ability in research comparable at least with that expected for the degree of Doctor of Philosophy. Blanks for application may be obtained from the School of Mathematics, Institute for Advanced Study, Princeton, New Jersey, and are returnable by February 1, 1948.

THE EIGHTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The eighth annual William Lowell Putnam Mathematical Competition, under the sponsorship of the Mathematical Association of America, will be held on Saturday, March 20, 1948. This Competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is open to undergraduates in the United States and Canada who have not received a degree.

The examination consists of two parts of three hours each. The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team or individual contestants may secure an application blank from Professor George Mackey, Hunt Hall 12, Harvard University, Cambridge 38, Massachusetts, by a postcard request. All applications must be filed with Professor Mackey not later than March 1, 1948. If three candidates are presented from a college or university, they are to constitute a team; if more than three are presented from any one college or university, the team of three must be named on the application.

The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to the rule may be made in cases of unusual necessity. Sealed copies of the examinations will be sent to the supervisor of the examination in time for the examination day and are not to be opened before the hour set. At the supervisor's first opportunity after the afternoon examination, the books are to be sent by registered mail or by express to Professor Mackey, who will forward them to a qualified reader chosen by the Association.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are $400, $300, $200 and $100 in the order of their rank. In addition, there will be prizes of $40, $30, $20 and $10 awarded to the members of these teams according to the rank of the team; a prize of $40 to

each of the five highest contestants and a prize of $20 to each of the succeeding five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to several teams next in order after the four winning teams and to the fifteen individuals next in order after the ten individual winners. For further encouragement of the Competition, there will be awarded at Harvard University (or at Radcliffe College in the case of a woman) an annual $1200 William Lowell Putnam Prize Scholarship to one of the first five contestants, this to be available either immediately or on the completion of the student's undergraduate work.

Reports on the seven previous competitions and examination questions will be found in this MONTHLY for May, 1938, 1939, 1940, 1941, 1942, October, 1946, and for August–September, 1947.

PERSONAL ITEMS

Assistant Professor C. F. Adler of New York University has been promoted to an associate professorship.

Dr. G. E. Albert of the Naval Ordnance Plant, Indianapolis, Indiana, has been appointed to an associate professorship at the University of Tennessee.

Adjunct Professor W. E. F. Appuhn of the Polytechnic Institute of Brooklyn has been appointed head of the department and professor of mathematics at St. Francis College, Brooklyn.

Assistant Professor B. H. Arnold of Montana State College has been appointed to an assistant professorship at Oregon State College.

W. H. Bradford, John McNeese Junior College of Louisiana State University, Lake Charles, Louisiana, has been promoted to an associate professorship.

Assistant Professor B. W. Brewer of Texas A. & M. has been appointed to an assistant professorship at Oregon State College.

Assistant Professor H. K. Brown of Lehigh University has been appointed to an assistant professorship of Mechanical Engineering at Northeastern University.

Ethel B. Callahan of Hartwick College, Oneonta, New York, has been promoted to a professorship.

Randolph Church of the United States Naval Postgraduate School has been promoted to a professorship.

Assistant Professor Mary Dean Clement of Wells College has been appointed to an assistant professorship at the University of Miami.

Professor A. B. Coble of the University of Illinois has been appointed visiting professor at Haverford College.

Dr. J. B. Coleman has been appointed to a professorship at Presbyterian College, Clinton, South Carolina.

Associate Professor H. L. Dorwart of Washington and Jefferson College has been promoted to a professorship.

Associate Professor E. D. Eaves of the University of Tennessee has been promoted to a professorship.

Professor L. R. Ford of the Illinois Institute of Technology was visiting professor at the University of Utah during June and July of the 1947 Centennial.

Professor Mario O. Gonzalez of Havana University has been appointed visiting professor at the University of Alabama.

Associate Professor Cornelius Gouwens of Iowa State College has been promoted to a professorship.

Assistant Professor Albert Grau of the University of Kentucky has been appointed to an associate professorship at the University of Alabama.

Professor L. M. Graves of the University of Chicago has been appointed to a visiting professorship at Indiana University for the current academic year.

William Gustin of the University of California at Los Angeles has been appointed to an assistant professorship at Indiana University.

D. W. Hall of the University of Maryland has been promoted to a professorship.

Dr. N. A. Hall of United Aircraft Corporation has been appointed to a professorship of thermodynamics at the University of Minnesota.

Dr. O. G. Harrold of Princeton University has been appointed to a professorship at the University of Tennessee.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The sixth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Pratt Institute, Brooklyn, N. Y., on Saturday, April 19, 1947.

There were one hundred and seventy-two present, including the following sixty-eight members of the Association: R. G. Archibald, L. A. Aroian, Aaron Bakst, Brother Bernard Alfred, Samuel Borofsky, C. B. Boyer, Benjamin Braverman, Paul Brock, A. B. Brown, Hobart Bushey, Jewell Hughes Bushey, Mannis Charosh, Ruth T. Coleman, T. F. Cope, W. H. H. Cowles, Constance H. Davis, D. R. Davis, J. N. Eastham, W. H. Fagerstrom, J. M. Feld, Edward Fleisher, R. M. Foster K. G. Fuller, Irving Gerst, B. P. Gill, Harriet M. Griffin, George Grossman, Frank Hawthorne, G. C. Helme, Morris Hertzig, E. Marie Hove, L. C. Hutchinson, Joseph Jablonower, H. S. Kieval, Helen L. Kutman, Nathan Lazar, C. H. Lehmann, A. A. LePori, M. E. Levenson, S. B. Littauer, H. F. MacNeish, May H. Maria, F. H. Miller, A. J. Mortola, M. A. Nordgaard, P. B. Norman, L. F. Ollmann, Max Peters, Walter Prenowitz, J. K. Reckzeh, Rose Roll, G. J. Ross, S. G. Roth, John Salerno, Charles Salkind, A. A. Schwartz, Aaron Shapiro, James Singer, J. E. Thompson, H. E. Wahlert, Etta A. Waite, Israel Wallach, Alan Wayne, Margaret C. Weeber, D. E. Whitford, John Williamson, J. M. Wolfe, H. J. Zimmerberg.

Professor W. H. H. Cowles and Mr. Morris Hertzig, Vice-Chairmen of the Section, presided at the morning and afternoon sessions respectively. At the opening of the morning session Dean Nelson S. Hibshman of the School of Engineering welcomed the Section to Pratt Institute. At the opening of the afternoon session a brief business meeting was held, with Professor H. E. Wahlert, Chairman of the Section, presiding. The following officers were elected for the coming year: Chairman, W. H. H. Cowles, Pratt Institute; Vice-Chairmen, Brother Bernard Alfred, Manhattan College, and George J. Ross, Erasmus Hall High School; Secretary, James Singer, Brooklyn College; Treasurer, Aaron Shapiro, Midwood High School. A report was presented by the Committee on Awards and Prizes. It was moved, seconded and carried without opposition that the Committee be continued for another year with a view to presenting its recommendations in writing, subject to the approval of the Executive Committee, with the call of the next annual meeting of the Section. At the close of the afternoon session all those present were invited to remain for tea at the Women's Club as guests of The Faculty Wives' Club of Pratt Institute.

The following papers were presented at the morning and afternoon sessions:

1. *The mathematics of magic squares*, by Harry Sitomer, New Utrecht High School, introduced by the Secretary.

The first n^3 positive integers can be written uniquely in the form $N=na+b$ where $a=0, 1, 2, \cdots, n-1$, and $b=1, 2, 3, \cdots, n$. Thus a magic square can be decomposed into two auxiliary squares, the first (called an A square) containing the a's, the second (called a B square) containing the b's; and corresponding a's and b's occupy those similarly-placed cells occupied by N. Any line in A (row, column, or diagonal) has a sum L_A, and a line in B has a sum L_B. Then $nL_A+L_B=n(n^2+1)/2$, the sum of each line in a magic square. Thus the problem of constructing magic squares is resolved through the construction of A squares and conjugate B squares. The simplest A square is regular, that is, $L_A=n(n-1)/2$. In regular B squares, $L_B=n(n+1)/2$. Often an A square can be transformed into a conjugate B square by adding one to each a and rotating the square 90°. This method of constructing magic squares simplifies the construction, permits a census of a set of squares, and leads naturally to methods for constructing magic cubes, hypercubes, and so forth.

2. *Elementary geometry as an algebraic system*, by Professor Walter Prenowitz, Brooklyn College.

An ordered linear geometry (for example, euclidean, ordered affine, or hyperbolic geometry) can be converted into an algebraic system by defining (1) $a+b$ for distinct points a, b to be the set of points between a and b (segment ab), and (2) $a+a$ to be a. The resulting system is a special type of multigroup or generalized group with many-valued composition. The difference $a-b$ of points a, b, defined as the set of points x for which $b+x$ contains a, is the prolongation of segment ab beyond a, provided $a\neq b$. A formalism is developed which enables one to derive geometrical properties algebraically. The fundamental classes of geometrical figures in non-metrical geometry, (1) convex sets, (2) linear spaces (points, lines, planes, $\cdots$), (3) half-spaces (rays, half-planes, $\cdots$) can be formulated respectively as semi-groups (subsets closed under $+$), subgroups (subsets closed under $+$, $-$) and residue classes of congruence relations.

3. *Mathematics in psychology*, by Dr. Lloyd Henry Beck, Department of Psychology, Yale University, introduced by the Chairman.

The design, execution, and analysis of psychological experiments involve the use of mathematics. The design requires hypothesizing correspondences between psychological elements and mathematical elements, and between psychological operations and mathematical operations. The execution obtains data relative to the hypothesis and requires measurement of the psychological elements. This measurement ranges from a statement of presence or absence at one extreme to quantification at the other. The execution frequently requires apparatus designed on the basis of mathematics in other sciences. The analysis of the data requires at a minimum description of association between variables, that may be described graphically or by an arbitrary function. In addition the design correspondences may enable one to deduce the experimentally-observed association at three levels: (1) the form of the function is given theoretically but the constants have to be determined by the data; (2) the form and some of the constants are given theoretically, the others being determined from the data; (3) both the form and the constants are given theoretically.

4. *What statistics, if any, in a required general mathematics course? Report of an attack on this problem at Queens College*, by Professor T. F. Cope, Queens College.

As a result of about eight years experience with a required mathematics course at Queens College, the speaker reported that the subject of statistics was of intrinsic interest to students in the arts and the social sciences, and it could, in his opinion, be used to great advantage by departments of mathematics in teaching mathematics to these students. He discussed in some detail the topics in statistics that are included in the work of the second term of a required one-year course at Queens College, and the methods used in presenting these topics.

5. *The tentative secondary school syllabus in mathematics for grades 7 through 12*, by Joseph Orleans, George Washington High School, introduced by the Secretary.

The tentative course of study in mathematics for grades 7 to 12, recently prepared by a committee for the New York State Department of Education under the chairmanship of the State Supervisor of Mathematics, must be considered in three parts: (1) the work of the seventh, eighth, and ninth years, (2) the geometry of the tenth year, and (3) the content of the eleventh and twelfth years. The work listed for the seventh, eighth, and ninth years resembles in general the content of what has come to be regarded as the mathematics of the Junior High School. It may involve a change in sequence of topics and shifting of emphasis; but it arouses no serious differences of opinion. The work of the eleventh and twelfth years likewise consists of the present half-year courses in intermediate algebra, trigonometry, advanced algebra, and solid geometry, with topics rearranged into two comprehensive one-year courses, with some deletions and some additions. It is the tenth year that may be the basis for serious differences of opinion among teachers of mathematics. In presenting the year of plane geometry, the committee proposes a new emphasis on the importance of definitions and of assumptions in mathematics and in non-geometric situations. The new course also calls for keeping alive throughout the tenth year the arithmetic and algebraic skills and concepts learned in previous years, and it includes the introduction of a short unit of coördinate geometry and a new emphasis on types of thinking in geometric and non-geometric situations.

6. *A report on high school mathematical preparation*, by Professor F. H. Miller, The Cooper Union School of Engineering.

The speaker reported on the results of a questionnaire, prepared by Professor S. G. Roth and himself, and circulated among fifty-seven high schools in the metropolitan New York area. Questions of varying degrees of difficulty, arranged under twelve headings on topics in algebra and

trigonometry, were listed. Chairmen of high school mathematics departments were asked to indicate whether these items were considered in elementary, intermediate, or advanced algebra courses, in trigonometry courses, or not at all. Results of the questionnaire, obtained from twenty-eight high schools, were presented and discussed by the speaker. The full questionnaire, distribution of replies received, and conclusions drawn from the results will appear in an early issue of the *Journal of Engineering Education*. The hope was expressed that similar projects will be undertaken in other regions so that college mathematics teachers may better determine the degree of preparation of their entering students.

C. B. BOYER, *Secretary*

APRIL MEETING OF THE KANSAS SECTION

The thirty-second annual meeting of the Kansas Section of theMathematical Association of America was held at the University of Wichita, in Wichita, on Saturday, April 19, 1947. Morning and afternoon sessions were held. Professor C. A. Reagan presided at these sessions. The morning session was a joint meeting with the Kansas Association of Teachers of Mathematics.

The attendance was one hundred fifty-seven including the following thirty-two members of the Association: Frances M. Breneman, Virginia L. Chatelain, L. E. Curfman, Paul Eberhart, H. P. Fawcett, W. H. Garrett, Edison Greer, J. R. Hanna, A. J. Hoare, W. C. Janes, L. E. Laird, C. F. Lewis, H. W. Linscheid, Anna Marm, Thirza Mossman, O. J. Peterson, P. S. Pretz, O. M. Rasmussen, C. B. Read, C. A. Reagan, L. M. Reagan, E. S. Robbins, R. G. Sanger, G. W. Smith, R. G. Smith, W. T. Stratton, Sister M. Helen Sullivan, C. B. Tucker, Gilbert Ulmer, E. B. Wedel, A. E. White, Ferna E. Wrestler.

At the business meeting the following officers were elected for next year: Chairman, Sister M. Helen Sullivan, Mt. St. Scholastica College; Vice-Chairman, R. G. Sanger, Kansas State College; Secretary-Treasurer, Anna Marm, Bethany College.

The following papers were presented:

1. *The development of teachers of mathematics*, by Professor Harold P. Fawcett, Ohio State University.

2. *Development of schools of mathematics*, by Professor R. G. Sanger, Kansas State College.

The development of certain schools of mathematics, such as the Pythagorean School, the one at the University of Paris, and the one at Göttingen, was considered. In addition, an attempt was made to ascertain the cultural, political, and social conditions amidst which mathematics might flourish.

3. *On non-euclidean planes*, by S. G. Kneale, University of Kansas.

In the euclidean plane, statements involving the notion of length follow from the three axioms to which the length is subjected. Thus, we arrive at the notion of a non-euclidean plane with a non-euclidean distance, where the last is defined by any numerically valued function of couples of points which satisfies these axioms. Many elementary configurations will change their shape in a geometry with a different definition for distance. Some radical and interesting changes will occur if the distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is defined as the greater of two numbers $|x_2-x_1|$ and $|y_2-y_1|$. The perpendicular bisector (that is the locus of points equidistant from two fixed points) is considered in this connection. In the plane with the distance defined as

above, the perpendicular bisector is in general a broken line whose shape depends on the relative position of the two points with respect to the coördinate axes. In two special cases, when the two points have the same x or y coördinate, the "perpendicular bisector" covers an entire area of infinite measure.

4. *Trigonometry as developed by use of vectors*, by F. B. Sleat, Kansas State College.

Elementary principles of vector analysis were employed to derive a great many of the formulae that are to be found in plane and spherical trigonometry.

5. *Pythagorean numbers*, by Professor G. W. Smith, University of Kansas.

Professor Smith exhibited a method which would yield all the primitive Pythagorean number triples, and then proved that for every right triangle whose sides form a Pythagorean number triple the radii of the inscribed and of the three escribed circles are integers.

ANNA MARM, *Secretary*

APRIL MEETING OF THE PACIFIC NORTHWEST SECTION

The first meeting of the newly organized Pacific Northwest Section of the Mathematical Association of America was held at the University of British Columbia, Vancouver, British Columbia, on April 10–11, 1947.

Forty-seven persons attended, including twenty-seven Association members, as follows: J. P. Ballantine, R. F. Bell, Daniel Buchanan, L. G. Butler, D. G. Chapman, C. L. Clark, C. M. Cramlet, Howard Eves, N. S. Free, W. H. Gage, F. L. Griffin, P. C. Hammer, H. H. Irwin, R. D. James, S. A. Jennings, J. R. F. Kent, M. S. Knebelman, A. S. Merrill, W. E. Milne, D. C. Murdoch, F. S. Nowlan, T. S. Peterson, A. R. Poole, J. J. Rowland, W. H. Simons, J. R. Vatnsdal, F. E. Wood.

The officers elected for next year were: Chairman, M. S. Knebelman, State College of Washington; Vice-Chairman, W. E. Milne, Oregon State College; Secretary-Treasurer, F. S. Nowlan, University of British Columbia. The Section voted to accept the invitation of the University of Oregon to hold the next meeting there in March, 1948.

Fifteen papers were presented at the sessions on Thursday afternoon and Friday morning, and a third meeting, devoted to an informal discussion of problems bearing on mathematical instruction in the Pacific Northwest was held on Thursday evening following a complimentary dinner given by the University of British Columbia for the visiting mathematicians and their guests. In addition, on Thursday afternoon, Professor S. A. Jennings, University of British Columbia, gave an interesting and instructive paper entitled *Topographical Methods in Algebra*.

The papers presented at the Thursday and Friday sessions are listed below:

1. *A formula of Liouville*, by Professor H. Gage, University of British Columbia.

The only published proof for formula (Q), stated by Liouville in the sixth of his famous eighteen articles, was given by E. T. Bell, who paraphrased an identity between elliptic functions.

Professor Gage presented an elementary proof. A general formula, which includes several of Liouville's results as particular cases, was also developed.

2. *Tertiary combination*, by Professor A. R. Poole, Oregon State College.

A set of postulates for a system with a single tertiary law of combination was given, and the fact that the tertiary combination cannot be obtained by suitably combining binary combinations was demonstrated.

3. *Representations of the Sylow subgroups of the symmetric group of degree p^n*, by Professor S. A. Jennings and D. G. Duncan, the latter introduced by Professor Jennings, University of British Columbia.

L. Kaloujnine has shown that the Sylow subgroups of the symmetric groups of degree p^n when p is prime may be represented as permutations of n dimensional vectors with coefficients in $GF(p)$. These representations were discussed and new results on the structure of these groups were obtained.

4. *On subgroup chains in a finite group analogous to the upper and lower central series*, by Professor D. C. Murdoch, University of British Columbia.

Various series of normal subgroups of a group G, analogous to the upper and lower central series, were discussed, in particular those series obtained by the use of such "commutators" as $a^s \cdot a^{-1}$ where s is an outer automorphism of G, and $(a^{s^{-1}}b^{s^{-1}})^s \cdot b^{-1}a^{-1}$ where s is an arbitrary permutation of the elements of G.

5. *Interference with a controlled process*, by Professor P. C. Hammer, Oregon State College.

In the terminology of statistical quality control a process is in control with respect to the production of a certain quality characteristic measured by a variable x if a sequence $x_0, x_1, x_2, \cdots, x_n$ of realizations of x might reasonably have been drawn at random from a fixed frequency distribution of x-values. If the occurrence of x_{k-1} results in a designed change of the distribution from which x_k is drawn, then the process is no longer in control. Letting a be a standard value desired for x, and m_k the mean of the distribution of x_k, Professor Hammer assumes interference in such a fashion that $m_k = m_{k-1} - c(x_{k-1} - a)$ where c is a positive number. The distribution of $\xi_k = x_k - m_k$, $k = 0, 1, 2, \cdots$, is assumed fixed, and the ξ_k are assumed to be mutually independent. This form of interference results in an increased standard deviation of x_k and a non-zero correlation between x_k and its predecessor x_{k-1}. The existence of an asymptotic distribution of x_k for $0 < c < 2$ is demonstrated.

6. *Successive approximations adapted to computation of complex roots*, by Professor W. E. Milne, Oregon State College.

A computational scheme is devised for practical calculation of irreducible quadratic factors of the form $s^2 + ps + q$ of a polynominal with real coefficients. The procedure is to approximate the real quantities p and q by a process of successive approximations. The procedure is devised with special reference to convenient computation with a calculating machine.

7. *On the numerical solution of linear integral equations*, by Professor G. Hacker, State College of Washington.

Linear integral equations of the first and second kinds are solved numerically by Gaussian quadrature in the form

$$\varphi(x) = f(x) + \lambda \sum_{i=1}^{n} A_i \cdot K(x, \mu_i)\, \varphi(\mu_i)$$

for $x=\mu_i$ where μ_i are the Legendre polynomial real zeros, and A_i the Gauss weight factors. Interpolation then provides the required numerical description of $\phi(x)$. Comparison of the method with those of Prasad and Bateman is made.

8. *Graphical solution of first order linear differential equations*, by Professor J. P. Ballantine, University of Washington.

The general equation discussed is $y'=[f(x)-y]/T(x)$, and y' is seen to be the slope of the line from any point (x, y) of the integral curve to $(x+T, f(x))$. The locus of the latter point can be drawn in advance, so that the tangent can be quickly drawn at any point of the integral curve. The solution is then obtained in the form of a graph which is really a series of straight line segments. Good results can be obtained, since each straight line segment is tangent to the curve at its midpoint and not at one of its ends. Special cases when $f(x)=0$, and $T(x)$ is a constant or a multiple of x are important. The method is very rapid for these cases.

9. *The general problem of stability of economic equilibria*, by Professor D. G. Chapman, University of British Columbia.

If a dynamic economic situation is represented mathematically by a set of equations involving time, then to determine the stability of an economic equilibrium so defined, it is necessary to study the behaviour of the solutions of this set of equations for large t. There is also a further question, namely, will the process of adjustment used by the individuals within the situation result in the same or any equilibrium point being attained? By the use of difference equations these two problems are studied, and it is shown for various economic situations that the problem of stability can be fully solved by answering both questions.

10. *Oscillating satellites with the force varying inversely as the nth power of the distance*, by Professor Daniel Buchanan, University of British Columbia.

The restricted problem of three bodies was discussed for the case when the force varies inversely as the nth power of the distance. The straight line and equilateral triangle equilibrium points were found, and the oscillating satellites near these points were obtained.

11. *Some related space curves*, by Professor Howard Eves, Oregon State College.

Let $x=x(s)$ be the parametric representation, in terms of the arc length s as parameter, of a given skew curve, and let α, β, γ be the unit vectors in the directions of the tangent, binormal, and principal normal respectively. The given curve may then be represented as $x=\int\alpha ds$. Associated with the curve are the two curves $x_2=\int\beta ds$, and $x_3=\int\gamma ds$. This speaker discussed some of the geometrical relations existing between the given curve and these related curves. In particular, certain linear combinations of the given curve and its related curves were considered. In addition to a number of new theorems, some known results were obtained in a novel way, for example, the general equations of all Bertrand curves and of all Mannheim curves were found.

12. *New methods in teaching polar coordinates in analytic geometry*, by Professor F. S. Nowlan, University of British Columbia.

Two essentially different systems of polar coördinates were considered. In the first, ρ is never negative, whereas in the second ρ is never positive. The replacements of ρ by $-\rho$ and θ by $\pi+\theta$ carry one from either system to the other. The methods developed enable one to avoid certain difficulties associated with the usual treatment of polar coördinates. Furthermore, it was shown that these methods have important applications to the calculus.

13. *The teaching of elementary calculus*, by Professor R. D. James, University of British Columbia.

Most of the calculus texts in use at present begin with a more or less systematic discussion of variables, functions, and limits, and then proceed to the topic of differentiation. It is Professor James' opinion that such a traditional beginning is unwise and that the most natural approach to the calculus is by way of the definite integral. He set forth reasons for this point of view and outlined the necessary steps in the alternative approach.

14. *A graphical treatment of the covariants of a binary cubic*, by Professor F. E. Wood, University of Oregon.

With three distinct points A, B, C of the complex plane, with complex coördinates a, b, c respectively are associated three points A', B', C' by making the cross ratios of (AB, CC'), (BC, AA'), (CA, BB') each equal to a given complex number α. Using all such α there is established a set Γ of point triples, associated with the given triple. In this paper are obtained the geometric relations between the triples of Γ and their relations to some of the covariants of the binary cubic form obtained from the cubic equation which has a, b, c as roots.

15. *New formulas for the numerical integration of* $y'=f(x, y)$, by R. J. Brown introduced by Professor F. L. Griffin.

It is assumed that the solution $y=y(x)$ of the differential equation $y'=f(x, y)$ has a valid Taylor series near the given point (x_0, y_0), from which three other points (x_1, y_1), (x_2, y_2), (x_3, y_3) at equal intervals, $\Delta x=h$, have been calculated initially to the desired accuracy. To predict a fourth such point, y_4 is taken as a linear function of y_0, y_1, y_2, y_3, and derivatives y_0', y_1', y_2', y_3'. The eight coefficients are determined so as to make the linear formula agree with the Taylor series through h^7, but the resulting formula is unsatisfactory because its large coefficients magnify the errors in the initially computed y_1, y_2, y_3. By omitting the terms in y_2 and y_0', however, a single infinitude of formulas is obtained, each of which is correct through h^4, some of which give results above the true y_4 and some below. Applications are made to some standard textbook examples. In one familiar case one formula gives eight decimal places correct when the initial computed values are obtained with sufficient accuracy.

F. S. NOWLAN, *Secretary*

CALENDAR OF FUTURE MEETINGS

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pennsylvania State College, Spring, 1948
ILLINOIS
INDIANA
IOWA, Fairfield, April 16–17, 1948
KANSAS, Atchison, April 10, 1948
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 24, 1948
OHIO
OKLAHOMA
PACIFIC NORTHWEST, Eugene, Oregon, March, 1948
PHILADELPHIA, Philadelphia, Pa., Nov. 27, 1948
ROCKY MOUNTAIN
SOUTHEASTERN, Charleston, S.C., March 19-20, 1948
SOUTHERN CALIFORNIA, Redlands, March 13, 1948
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, Schenectady, N. Y., May 1, 1948
WISCONSIN, Beloit, May 8, 1948

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R. S. Underwood is Professor of Mathematics and Astronomy at Texas Technological College. T. R. Nelson is Associate Professor of Mathematics at the Agricultural and Mechanical College of Texas. S. Selby is Professor of Mathematics and Head of the Department at the University of Akron.

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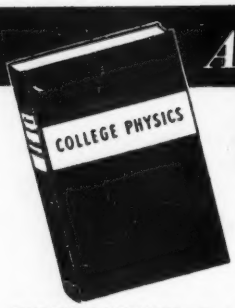

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